How to use Excluded Middle Safely

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How to reason relative to a principle in a foundation that invalidates it?

“For all” vs. “for every instance”

Reasoning principles as universally quantified statements:

LEM	$\forall p : \Omega .\, p \lor \neg p$
LPO	$\forall f : 2^{\mathbb{N}} .\, (\exists n .\, fn = 1) \lor (\forall n .\, fn = 0)$
ALPO	$\forall x : \mathbb{R} .\, x \leq 0 \lor x > 0$
CT	$\forall f : \mathbb{N}^{\mathbb{N}} .\, \exists n : \mathbb{N} .\, \forall i .\, \exists j .\, T(n, i, j) \land U(j) = f(i)$
AC	$\forall R \subseteq A \times B .\, (\forall a .\, \exists b .\, a\,R\,b) \Rightarrow \exists f : B^A .\, \forall a .\, a\,R\,(f\,a)$

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We could use *schemata*, but such meta-theoretic devices are not easily formalized.

We shall use *modalities* instead.

Oracle modality

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Examples:

- excluded middle oracle $\mathsf{dec} : \Omega \to \Omega$ with $\mathsf{dec}\,s := s \vee \neg s$,
- classical oracle for $S \subseteq \mathbb{N}$ is $\lambda n{:}\mathbb{N}.\,(n \in S \vee n \notin S)$.

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Define the **oracle modality** $\circledcirc_P : \Omega \to \Omega$ inductively as:

$$\circledcirc_P\, s \quad := \quad s \lor \exists a{:}A.\,(P\,a \Rightarrow \circledcirc_P\, s).$$

"Prove s or ask $a : A$ and proceed upon oracle answering $P\,a$."

Construction of $\mho_P$

In a topos we may construct $\mho_P$ as the least fixed-point of $r \mapsto (s \vee \exists\, a : A.\, P\, a \Rightarrow r)$:

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In type theory, $\mho_P$ is a higher-inductive type with constructors:

- `prove` : $s \longrightarrow \mho_P\, s$,
- `ask` : $(a : A) \longrightarrow (P\, a \longrightarrow \mho_P\, s) \longrightarrow \mho_P\, s$,
- propositional truncation.

Properties of $\mathbb{O}_P$

- The modality laws hold:

$$s \leq t \Rightarrow \mathbb{O}_P\, s \leq \mathbb{O}_P\, t \qquad \text{and} \qquad s \leq \mathbb{O}_P\, s = \mathbb{O}_P\, (\mathbb{O}_P\, s).$$

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- $\bigcirc_P \le \bigcirc_Q$ if, and only if, $\forall a{:}A.\, \bigcirc_Q\,(P\,a)$.
- If $\neg P\,a$ for some $a : A$ then $\bigcirc_P = \lambda\_.\, \top$ is trival.
- $\bigcirc_P$ may be non-trivial even if $\neg\forall a{:}A.\, P\,a$ holds:
  - Define $\mathsf{lpo} : 2^{\mathbb{N}} \to \Omega$ with $\mathsf{lpo}\, f := (\exists n.\, fn = 1) \vee (\forall n.\, fn = 0)$.
  - The effective topos validates $\neg\forall f{:}2^{\mathbb{N}}.\, \mathsf{lpo}\, f$.
  - $\bigcirc_{\mathsf{lpo}}\varphi$ means “φ is realized relative to the Halting oracle”.

A couple of theorems

Theorem

Every modality is an oracle modality.

Proof. Modality $j : \Omega \to \Omega$ equals $\bigcirc_P$ for $P : (\sum(s : \Omega).\, j\, s) \to \Omega$, $P\,(s, \_) := s$. $\square$

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Theorem

Let $P_i : A_i \to \Omega$ be a family of oracles. The supremum of $\circledcirc_{P_i}$'s is $\circledcirc_Q$ where $Q : (\sum(i : I).\ A_i) \to \Omega$ with $Q(i, a) := P_i\, a$.

Open modality vs. oracle modality

- For $p : \Omega$ the **open modality** $\bigcirc_p : \Omega \to \Omega$ is $\bigcirc_p\, s := (p \Rightarrow s)$.
- Always $\mathcal{O}_P \leq \bigcirc_{\forall a : A.\, P\, a}$.

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- $\bigcirc_{\forall a : A.\, P\, a} \leq \circledcirc_P$ if, and only if,

$$\circledcirc_P(\exists\, a_1, \ldots, a_n : A.\, P\, a_1 \wedge \cdots \wedge P\, a_n \Rightarrow \forall\, a : A.\, P\, a)$$

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We often state $\bigcirc_{\forall a : A.\, P\, a}\, \varphi$, but then give a proof that factors through $\circledcirc_P\, \varphi$.

Provocation

In Type Topology library, all proofs of $\bigcirc_{\mathsf{LEM}}\, \varphi$ factor through $\circledcirc_{\mathsf{dec}}\, \varphi$.

(Where $\mathsf{dec}\, p := p \vee \neg p$ and $\mathsf{LEM} := \forall\, p : \Omega.\, \mathsf{dec}\, p$.)

Sheaves for oracle modalities (preliminary)

Consider an oracle $P : A \to \Omega$.

The **sheafification** $\mathsf{sh}_P\, X$ of a type X is the higher-inductive type:

- $\mathtt{ret} : X \to \mathsf{sh}_P\, X$
- $\mathtt{ask} : \prod(a : A).\, (P\, a \to \mathsf{sh}_P\, X) \to \mathsf{sh}_P\, X$
- $\mathtt{equ} : \prod(a : A).\, \prod(t : P\, a \to \mathsf{sh}_P\, X).\, \prod(u : P\, a).\, \mathtt{ask}\, a\, t = t\, u$

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X is an $\bigcirc_P$-sheaf when there is $h : \mathsf{sh}_P\, X \longrightarrow X$ satisfying $h\,(\mathtt{ret}\,x) = x$.

Think of h as a *handler* extracting an element from the leaf of a computation tree.

Further & Related Work

Further work:

- Get the definition of sheaves correct.
- What about *non-propositional oracles* $P : A \longrightarrow \mathsf{Type}$? Example:

$$\mathsf{choose} : \mathsf{Type} \longrightarrow \mathsf{Type}$$
$$\mathsf{choose} : A \mapsto (\|A\|_{-1} \longrightarrow A)$$

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Related work:

- E. Rijke, M. Shulman, B. Spitters: *Modalities in homotopy type theory*, Logical Methods in Computer Science, January 8, 2020, Volume 16, Issue 1.
- Andrew W. SWan: *Oracle modalities*, arXiv:2406.05818, June 2024.
- Takayuki Kihara: *Lawvere-Tierney topologies for computability theorists*, Transactions of the American Mathematical Society, Series B, 10 (2023), 48–85.