Towards synthetic locale theory

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Types and Topology
A workshop in honour of Martín Escardó's 60th birthday

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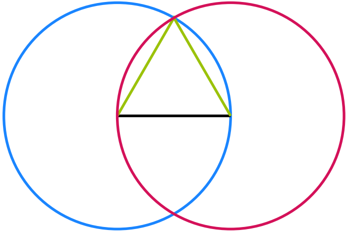
Some memories

- I think I first became aware of Martín's work at the MFO workshop on Mathematical Logic in November 2011, via Paulo Oliva's talk on *Bar Recursion and the Product of Selection Functions*.
- Around the same time, I noticed many thoughtful messages from Martín on the `agda`, `constructivenews` and `HomotopyTypeTheory` mailing lists.
- The first time I remember meeting Martín was at the Workshop on Homotopy Type Theory in Bonn, February 2016.
- Since then I've been very inspired both by Martín's earlier work on topology (following Smyth's dictionary) and the later work on univalent type theory.
- This talk is in part about combining these strands . . .
  Types and *Topology*

Synthetic mathematics

. . . large parts of modern mathematical research are based on a dexterous blending of axiomatic and constructive procedures. *(Weyl 1985, written ca. 1953–1955)*

The primordial example of synthetic mathematics is the Euclidean geometry of basic figures, that are *postulated* rather than *analyzed* in terms of something else/more primitive, and whose basic properties are taken as *axioms*.



(Another closely related idea: *purity of methods*)

Russell said this has all the benefits of theft over honest toil! But the toil of *constructing* models repays what was stolen.

Modern examples of synthetic mathematics

Modern examples:

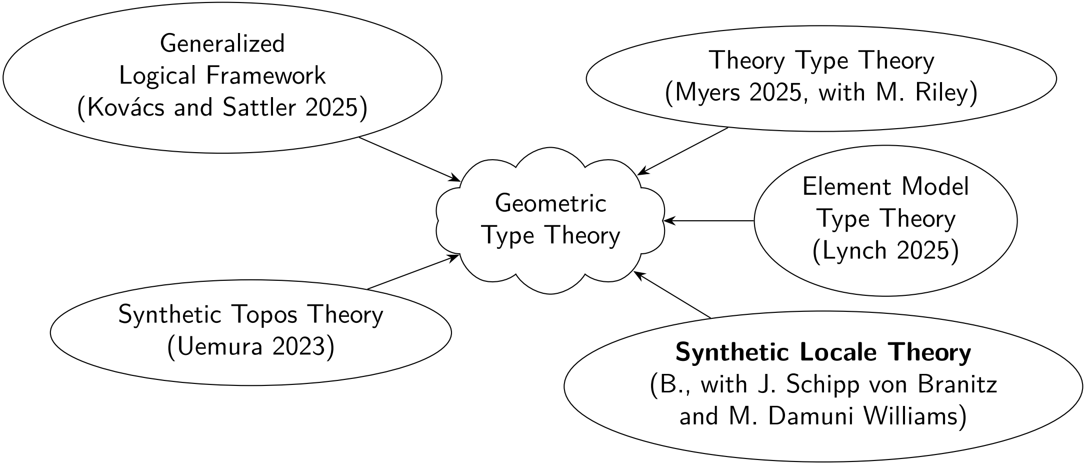
- Synthetic differential geometry: The differential line R is postulated as $\mathbb{Q}$-algebra with the Kock–Lawvere axiom
$$R[x]/(x^2) \xrightarrow{\sim} (\{\, d : R \mid d^2 = 0 \,\} \to R)$$
- Synthetic algebraic geometry: The algebraic line R is postulated as a local ring satisfying
$$A \xrightarrow{\sim} (\mathrm{Spec}(A) \to R),$$
where $\mathrm{Spec}(A) := \mathrm{Hom}_R(A, R)$, for all f.p. R-algebras A. Also: Zariski local choice.
- Synthetic computability theory: Assume Church's thesis
- Synthetic domain theory, recently Pugh and Sterling (2025): Assume partial map classifier Σ, a bounded distributive lattice with Phoa's principle
$$(\Sigma \to \Sigma) \xrightarrow{\sim} \sum_{x,y:\Sigma} x \leq y$$

More examples of synthetic mathematics

- Synthetic (higher) category theory: Extends synthetic domain theory with Σ totally ordered, and the universe generated by the simplices $\Delta^n \subseteq \Sigma^n$. (Riehl and Shulman 2017; Gratzer, Weinberger, and Buchholtz 2024; Gratzer, Weinberger, and Buchholtz 2025)
  In forthcoming work we define the $(\infty, 1)$-category of $(\infty, 1)$-categories.
- Synthetic Stone duality (Cherubini et al. 2024): Use the boolean algebra 2 with the Stone duality axiom:
  $$B \xrightarrow{\sim} (\mathrm{Spec}(B) \to 2)$$
  for all countably presented boolean algebras B, where $\mathrm{Spec}(B) := \mathrm{Hom}(B, 2)$. (Plus local choice and two more axioms.)
- Synthetic real cohesion via modal type theory (Shulman 2018)
- Synthetic measure theory, probability theory, quantum theory, ...

That's all very nice, but we'd like to relate the internal results to the external, make combinations (simplicial + cohesion, etc.), go between models, and more.

Towards geometric type theory



(All works in progress)

Synthetic Mathematics eats itself

Our goal is to build towards a *synthetic mathematics of synthetic mathematics*: Postulate constructions of theories, and give axioms that allow us to derive the "local" axioms for each theory, following Blechschmidt (2023), *general Nullstellensätze*.

Final intended model is the multiverse of toposes (Blechschmidt and Oldenziel 2023) (extending the set-theoretic multiverse), with distinctions between *étale maps* and *all geometric maps*, and corresponding local modal machinery to keep track of *arbitrary* and *geometric constructions* (Σ, $=$, finitary HITs), and to allow us to move freely between the *internal* and the *external*.

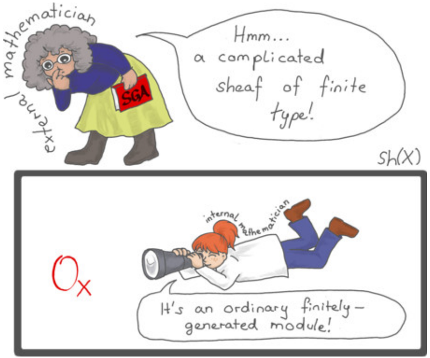


Illustration: Carina Willbold (CC BY-SA)

Challenges for Geometric Type Theory

The idea would be to have judgments (cf. EMTT):

- X is a topos/theory
- $x : X$ is a point/model
- A is étale/a type
- $a : A$ is a section/an element

(plus equality etc.)

Then need "local modalities", perhaps using refinement of MATT (Shulman 2023), plus bespoke rules ...

Some challenges:

- The intended model, Topos is not a model of type theory, not locally cartesian closed, etc.
- It's not remotely locally small, maps $X \to [T]$ are models of T in $\mathrm{Sh}(X)$, which are unbounded (e.g., $T = \mathrm{Ring}$).
- Usual strategy would be to stratify via an (external) universe hierarchy $\mathcal{V}_i$. Then Topos is enriched in stratified types?
- ...

Synthetic Locale Theory

Our approach is to start with the simplest case of *small, propositional* (geometric) theories, presenting locales, and *small* maps. If $\mathcal{V}_i$, $i = 0, 1, \ldots$, are the external universes, consider the site Loc_0 of $\mathcal{V}_0$-presented locales and $\mathcal{V}_0$-sized maps. (In general, $[T] \to_i [S]$ are models of S in $\mathcal{O}_i[T] := \mathrm{Sh}_i^{\leq -1}[T]$. We're working predicatively in anticipation of going higher.)

There's a natural (locally $\mathcal{V}_0$-small) étale topology on Loc_0 given by the axioms of the codomain theory.

In $\mathrm{Sh}(\mathrm{Loc}_0)$ we have a model of type theory, with a usual university hierarchy $\mathcal{U}_i$, $i \geq 1$, coming from $\mathcal{V}_i$. We have a split-context calculus to accommodate $\flat$. (Not covered by MTT!)

We define $\mathcal{U}_0 := \Delta(\mathcal{V}_0)$, the universe of locally constant sheaves, and $\mathbb{S}_0 := \mathrm{y}(\mathbb{S})$. These only have geometric constructors.

We have that $\mathbb{S}_0$ is a distributive lattice with $\mathcal{U}_0$-sups ($\leq : \mathbb{S}_0 \to \mathbb{S}_0 \to \mathcal{U}_1$).

Duality

Theorem

In this model we have internally: For any $\mathcal{U}_0$-theory T, there is a QIT giving an $\mathbb{S}_0$-algebra $A := \mathbb{S}_0[T]$ such that evaluation is an isomorphism

$$A \to (\mathrm{Spec}(A) \to A),$$

where $\mathrm{Spec}(A) := \mathrm{Hom}_{\mathbb{S}_0}(A, \mathbb{S}_0) \simeq \mathrm{Mod}_T(\mathbb{S}_0) := [T]$.

This is the starting point for doing locale theory synthetically.

To recover the non-geometric connective/constructions, we have for *crisp* theories T:

$$\flat(\mathrm{Prop}_1^{[T]}) \to \flat(\mathbb{S}_1^{[T]}),$$

where $\mathbb{S}_1$ another universe. (We're still working out whether this is postulated or defined ...)

Outlook

Immediate next steps:

- Prove more theorems of formal topology synthetically (and maybe that Cantor space is searchable!),
- Derive quasicoherent induction from duality
- (Finish the paper!)
- Formalization in Agda (with $\flat$)

Further, think about integrating other ideas:

- Improve the model:
  - Going higher: $(n, 1)$-toposes, $n \leq \infty$
  - Going larger: larger theories and/or larger maps internalized
- Improve treatment of iterated internalization (with inspiration from GATs, Arithmetic Universes, and locally presentable categories)
- Perhaps this will make the synthetic locale theory look more like the analytic one?
- Devise a geometric type theory (with normalization, etc.)

Happy Birthday, Martín!

- Blechschmidt, Ingo (2023). *A general Nullstellensatz for generalized spaces*. Draft. URL: https://rawgit.com/iblech/internal-methods/master/paper-qcoh.pdf.
- Blechschmidt, Ingo and Alexander Gietelink Oldenziel (2023). *The topos-theoretic multiverse: a modal approach for computation*. URL: https://www.speicherleck.de/iblech/stuff/early-draft-modal-multiverse.pdf.
- Cherubini, Felix, Thierry Coquand, Freek Geerligs, and Hugo Moeneclaey (2024). *A Foundation for Synthetic Stone Duality*. arXiv: 2412.03203 [math.LO].
- Gratzer, Daniel, Jonathan Weinberger, and Ulrik Buchholtz (2024). *Directed univalence in simplicial homotopy type theory*. arXiv: 2407.09146 [cs.LO].
- — (2025). *The Yoneda embedding in simplicial type theory*. arXiv: 2501.13229 [cs.LO].
- Kovács, András and Christian Sattler (2025). *A Generalized Logical Framework*. Presentation at TYPES 2025. URL: https://msp.cis.strath.ac.uk/types2025/abstracts/TYPES2025_paper62.pdf.

- Lynch, Owen (2025). *Element Model Type Theory*. Oxford Seminar, Topos Institute. URL: https://youtu.be/Id-9XE5TsA8?list=PLhgq-BqyZ7i4zZ9nAIcTQ3w6H0YBo8DR0.
- Myers, David Jaz (2025). *Liberating synthetic quasi-coherence from forcing*. Based on joint work with Mitchell Riley. URL: https://topos.institute/blog/2025-07-13-liberating-synthetic-quasi-coherence-from-forcing/.
- Pugh, Leoni and Jonathan Sterling (2025). "When is the partial map classifier a Sierpiński cone?" In: *2025 40th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pp. 718–731. DOI: 10.1109/LICS65433.2025.00060.
- Riehl, Emily and Michael Shulman (2017). "A type theory for synthetic ∞-categories". In: *Higher Structures* 1 (1), pp. 147–224. DOI: 10.21136/HS.2017.06.
- Shulman, Michael (2018). "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Math. Structures Comput. Sci.* 28.6, pp. 856–941. ISSN: 0960-1295,1469-8072. DOI: 10.1017/S0960129517000147.
- — (2023). "Semantics of multimodal adjoint type theory". In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 3-Proceedings of... ISSN: 2969-2431. DOI: 10.46298/entics.12300.
- Uemura, Taichi (2023). *Synthetic topos theory*. URL: https://uemurax.github.io/synthetic-topos-theory/.

Weyl, Hermann (1985). “Axiomatic versus constructive procedures in mathematics”. In: *The Mathematical Intelligencer* 7.4, pp. 10–17. ISSN: 0343-6993. DOI: 10.1007/bf03024481.