Diagram chasing in the Coq-HoTT library

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Outline:

- Diagram chasing and past work
- A new, pragmatic approach

These slides and a link to the github repo are available at:

https://jdc.math.uwo.ca/papers.html

Introduction

Martín Escardó's work is full of examples of studying a simple-sounding problem and getting a beautiful theory.

I'll highlight his work on injective types and on compact / searchable types as being great examples of this, and we've also heard about many other examples during this event.

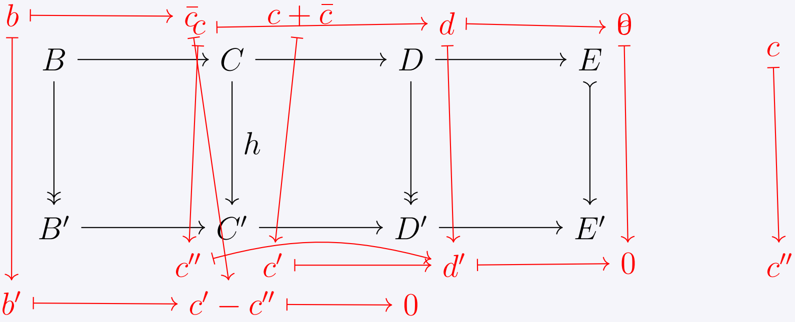
This talk is similar in that it starts with a simple-sounding problem.

But instead of a beautiful theory, we'll just end up with a pragmatic tool!

Diagram chasing

Many results about abelian groups, R-modules, and more general objects can be proved using the technique of diagram chasing. For example:

Proposition. Given a commuting diagram of abelian groups



with exact rows and with the indicated maps being epimorphisms and monomorphisms, the map h is an epimorphism.

Abelian categories and more

We'd like to be able to use the diagram chasing technique in categories whose objects don't have "elements".

We're particularly interested in abelian categories, but have formalized our approach with much weaker assumptions:

- We assume that our category is enriched over abelian groups,
- and that pullbacks of epimorphisms exist and are epimorphisms.

Examples include any abelian category (e.g. abelian groups, R-modules, sheaves of abelian groups or modules, abelian group objects in a topos, categories of left exact functors, etc.) as well as categories such as torsion-free abelian groups.

A lot of work has been done to try to generalize diagram chasing to all abelian categories.

Past approaches to diagram chasing

- Using generalized elements $P \to A$ as the elements of A does not work, because such an element doesn't lift through an epimorphism $B \twoheadrightarrow A$.
- Mac Lane proposed an approach using certain equivalence classes of generalized elements, which he called members.

  The equivalence relation is generated by $(P \to A) \sim (P' \twoheadrightarrow P \to A)$.

  He proved two standard diagram chasing lemmas with his approach, but left our example as an exercise. As far as I can, it can't be done using his set-up!

  More generally, his set-up does not handle differences and sums very well. In fact, in his set-up, $x = -x$ for every member!

  This is the approach used by Markus Himmel (mathlib in Lean) and Tomi Pannila (Unimath in Coq).
- There are various theorems that say that a nice enough abelian category has an exact embeding into a category of R-modules, e.g. the Freyd-Mitchell embedding theorem. But the proofs of these results are not constructive (A.G. Montaruli, 2024).

Lifting generalized elements

We use generalized elements and Mac Lane's equivalence relation

$$(P \to A) \sim (P' \twoheadrightarrow P \to A)$$

but we keep track of generalized elements, not their equivalence classes.

Suppose we are given a generalized element b in the codomain of an epimorphism:

$$\begin{array}{ccc} P' & \xrightarrow{a} & A \\ {\scriptstyle e}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow{\scriptstyle f} \\ P & \xrightarrow{b} & B \end{array}$$

We can form the pullback to get a generalized element a in A.
Note that $f\,a \sim b$, so up to $\sim$ we have lifted b through f.

Definition. Given $b : P \to B$ and $f : A \to B$, the type of **Mac Lane lifts of b through f** is

$$\mathrm{Lift}\ b\ f :\equiv \sum_{P'} \sum_{e:P' \twoheadrightarrow P} \sum_{a:P' \to A} (f\,a = b\,e).$$

Lifting generalized elements, II

Definition. Given $b : P \to B$ and $f : A \to B$, the type of **Mac Lane lifts of b through f** is

$$\text{Lift}\ b\ f :\equiv \sum_{P'} \sum_{e:P' \twoheadrightarrow P} \sum_{a:P' \to A} f\,a = b\,e.$$

Mac Lane showed the following facts:

Lemma. A map $f : A \to B$ in an abelian category is epi if and only if every generalized element b in B has a Mac Lane lift through f.

Lemma. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if every generalized element b in B with $g\,b = 0$ has a Mac Lane lift through f.

There are similar characterizations of monics, isomorphisms, and zero maps.

Bergman's approach

As mentioned earlier, Mac Lane's approach cannot in general add or subtract generalized elements.

In unpublished work, G. Bergman suggests using the above procedure, and keeping track of the chain of epimorphisms that arises by doing repeated lifts:

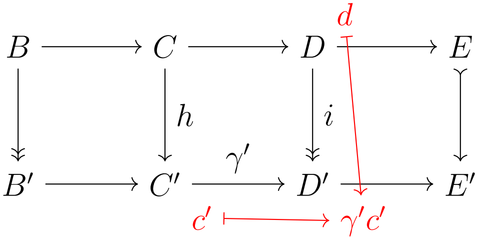
$$\cdots \twoheadrightarrow P_2 \twoheadrightarrow P_1 \twoheadrightarrow P.$$

This works, but requires bookkeeping. This is the approach used by Joël Riou (mathlib in Lean).

We found a way to make it just as smooth as classical diagram chasing.

I'll illustrate by going back to our first example.

Our approach, by example



Context (abbreviated):
Rows exact. Indicated arrows epi and mono.

Goal:
Lift c' h

$c' : P \to C'$
$d : P \to D$
$p : i\,d = \gamma'\,c'$

Summary: by using a lemma about Lift, generalizing, and renaming, it appears that $\gamma'\,c'$ has a lift d with the same domain P.

Our approach, theory

This simple lemma is all we require:

Lemma. For $P' \xrightarrow{e} P \xrightarrow{a} A$ and appropriate f, g and h:

- Lift $(a\,e)$ h if and only if Lift a h.
- $f\,a = g\,a$ if and only if $f\,a\,e = f\,a\,e$.

We have implemented a short tactic (34 lines) in Coq-HoTT that uses these results to make it appear that all elements have lifts without changing P.

One direction of each statement is used in hypotheses and the other in goals to ensure that each generalized element a appears precomposed with e.

Then we generalize $a\,e$ to eliminate all occurences of e and P.

Since, at any moment in the proof, all generalized elements in an object A have the same domain P, we can add and subtract such elements without any trouble, and the usual laws hold, such as $f(a + a) = f\,a + f\,a'$.

Examples

We have shown this method to be robust by formalizing a number of standard results in homological algebra:

- The five-lemma.
- Various forms of the 3×3 lemma.
- The spider lemma, the snail lemma, the small baby dragon lemma, the diamond lemma, and other lemmas that don't have cute names.

The proofs of these results are exactly parallel to the classical proofs.

In particular, no propositional truncations need to be dealt with!

The lifts exist purely rather than merely, so the proofs are simpler than the proofs would be for abelian groups in HoTT.

In addition, univalence and function extensionality are not used at all for the formal set-up or the proofs.

For function extensionality, we get away with this by using homotopies between functions instead of equalities.

Formalization

https://github.com/jdchristensen/HoTT-diagram-chasing

Formal set-up (~700 loc), homological algebra (~765 loc), misc (~900 loc).

Limitations.

- We use a tactic to hide the details, and I don't know if the approach can be done without tactics in Agda, Martín's proof assistant of choice.
- In general, it is hard in all frameworks to define a map using some kind of abstract elements.

  We hope to use the sequence of epimorphisms to extend the technique to allow this.

  J. Arnoult, A. Lafont, A. Mahboubi and M. Piquerez have work in progress that aims to go further than what I have described here. In addition to detecting exactness, epimorphisms, etc., their approach will let you use elements to construct maps and check equality of maps.

References

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