Sheaf models of type theory

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Types and Topology; a workshop in honour of Martín Escardó 60th birthday

Meeting Martín

I first met Martín during a Dagstuhl meeting June 2000; very nice evening discussion where Martín explained to me some of his work on the patch topology.

Discussions in airports, busses, Princeton, Oslo, ...

Thanks for all these nice interactions!!

In particular, multiple discussions on sheaf models and choice sequences.

How to combine dependent type theory and choice sequences?

Presheaf and Sheaf models

The notion of presheaf and sheaf models is relatively recent.

But it already has a rich history and appears in very different contexts:

-Beth (1955), Kripke (1958) formal model of intuitionistic logic, *temporal* and *epistemological* intuitions

-Eilenberg-Zilber (1950) combinatorial representations of spaces with simplicial sets, *spatial* intuitions

-Grothendieck (1960), big Zariski topos, *spatial* intuitions

The work I will present mixes these various intuitions.

Sheaf models of choice sequences

Usually presented with Baire spaces: sequences of natural numbers.

Maybe better to first look at binary sequences, using Cantor space instead, here denoted by $C = 2^{\mathbb{N}}$.

One intuition: working with sheaf model over C corresponds to adding one "generic" sequence α (D. Scott?).

Intuitively this generic sequence α behaves like an *unknown but fixed* infinite sequence, allowing us to reason about all sequences simultaneously.

Similar to the "small" Zariski topos over a ring, where we add one "generic" prime ideal of the ring.

Sheaf models of choice sequences

This is a powerful technique for showing uniform continuity as an *admissible* principle.

Let $f : 2^{\mathbb{N}} \to 2$ be a function defined in a type theory.

If we apply this function to the generic sequence α, we get a tree which witnesses that the functional f has to be uniformly continuous.

The tree is like an oracle that encodes how the functional f interacts with the function α (cf. joint work with Guilhem Jaber and Martín's paper on effectful forcing).

Sheaf models of choice sequences

But this does not give a model where uniform continuity holds *internally*.

Applying f to α only shows continuity as an *admissible* principle; to get continuity internally, we should be able to generate new independent generic sequences.

Intuitively the functional f may depend on the generic sequence α and we need to apply f to a *new* independent generic sequence.

Sheaf models of choice sequences

We want to capture the fact that at any moment of time, we can have access to a new choice sequence.

This is achieved by working with a *category* of spaces $X, Y, \ldots$, where the projection map $X \times C \rightarrow X$ corresponds to the addition of a new sequence.

If $X = X_0 + X_1$ then the injection map $X_0 \rightarrow X$ corresponds to the acquisition of new knowledge about the choice sequence.

In general a map $Y \rightarrow X$ corresponds to a change of knowledge.

Sheaf models of choice sequences

The two maps $X_0 \rightarrow X$ and $X_1 \rightarrow X$ form a covering: eventually we should get the knowledge $X_0 \rightarrow X$ or the knowledge $X_1 \rightarrow X$.

This defines a coverage on this category of spaces, hence a Grothendieck topology.

In the corresponding sheaf models, uniform continuity holds internally.

Sheaf models of choice sequences

One can check that, in this model, 2 is the following sheaf:

$$2(X) = \text{set of clopen subsets of } X\text{, which is } \mathsf{Cont}(X, 2)$$

$2^{\mathbb{N}}$ is *represented* by C

$$\mathsf{Cont}(X, C) = \mathsf{Cont}(X, 2^{\mathbb{N}}) = 2(X)^{\mathbb{N}}$$

So $F^{2^{\mathbb{N}}}(X) = F^{Yo(C)}(X)$ is $F(X \times C)$ for any sheaf F

In particular, if B is the Boolean algebra of clopen subsets of C we get that $B \to 2^{2^{\mathbb{N}}}$ is an isomorphism.

Internally, functions $2^{\mathbb{N}} \to 2,\ \ 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ are uniformly continuous.

Uniform treatment of uniform continuity

Using the internal languages of toposes in algebraic geometry
Ingo Blechschmidt Ph.D. thesis, 2017

Suggests an axiomatic system for the big Zariski topos, with R local ring satisfying the

Duality principle: *the canonical map* $A \rightarrow R^{Sp(A)}$ *is an* isomorphism

where $Sp(A) = Hom(A, R)$ for R-algebra, and A is finitely presented on R.

Ingo notices that we can deduce *internally* several properties of the Zariski topos from this axiom. A similar remark appears in the work of Anders Kock with his *comprehensive* axiom.

Uniform treatment of uniform continuity

Uniform continuity can be seen as an instance of the duality principle!

If B free Boolean algebra on countably many elements $b_0, b_1, \dots$ then

$$Sp(B) = Hom(B, 2) = 2^{\mathbb{N}}$$

So duality is exactly in this case that the canonical map $B \to 2^{Sp(B)}$ is an isomorphism.

This expresses internally that any function $2^{\mathbb{N}} \to 2$ is uniformly continuous.

Uniform treatment of uniform continuity

In the big Zariski topos, the duality principle implies that the canonical map

$$R[X_1, \ldots, X_n] \to R^{R^n}$$

is an isomorphism.

All functions $R^n \to R$ are *polynomials*.

Sheaf models and type theory

(What I learnt from Martín and Chuangjie Xu.)

While type theory has a natural interpretation in presheaf models, to extend this interpretation to *sheaf* models works for interpreting Π, Σ, but it is problematic for *universes*.

For a sheaf model over a space X, we may try to define $\mathcal{U}(V)$ as the set of sheaves over the open subset V, but there is a problem with the patching condition: uniqueness holds only *up to isomorphism*, not up to strict equality.

This issue was known and discussed by Grothendieck's school, see e.g. EGA 1, 3.3.1. It was a motivation for introducing the notion of *stacks*, with more subtle patching conditions.

Sheaf models and type theory

Similar questions appear when one wants to patch together *structures*, as opposed to *elements* (e.g. value of a function), e.g. in Weil's definition of a manifold.

We want to consider "groupoid-valued" sheaves: stacks.

But then also 2-groupoid-valued, $\ldots$ and ∞-groupoid-valued sheaves.

The goal of the first part of the letter from Joyal to Grothendieck, 1984, was precisely to build such a model of ∞-groupoid-valued sheaves.

However, this is done in a non constructive way (crucial use of Barr's Theorem, using the Quillen model structure on simplicial sets).

Sheaf models and type theory

An internal treatment of higher stacks and derived schemes is also desirable; it would probably rest upon a version of homotopy type theory as the internal language of ∞-toposes.
Ingo Blechschmidt's Ph.D. thesis, 2017

This is what we have been doing with a group of postdocs and PhD students.

In particular, discovery of a *new* axiom/principle (question raised in Ingo's thesis, is duality principle enough?).

Local choice: *Any surjective map into a space has locally a section*

Sheaf models and type theory

Three different examples, that can be developed axiomatically.

(1) big Zariski ∞-topos, access to the elegant definition of cohomology groups in homotopy type theory, and definition of objects such that the type of lines or theorems such as: affine schemes are closed by dependent sum types

(2) light condensed stacks, with an internal version of Dyckhoff's Theorem (Čech computation of cohomology of compact Hausdorff space) or theorems such as: Stone spaces are closed by dependent sum types (thanks Martín!)

(3) ∞-topos of non trivial bounded and linear distributive lattice, model of directed univalence

Sheaf models and type theory

Thanks to the work of Jonas Höfer and Christian Sattler, we now have constructive models of all these examples.

These models are defined in a weak meta theory.

Surprisingly, one key step is to be able to define the right *presheaf* model.

The topology/site is elegantly handled *internally* using left exact modalities.

Sheaf models and type theory

In summary, the goal of understanding better models of choice sequences has led to unexpected developments, mixing ideas from intuitionistic logic, algebraic geometry and homotopy theory.

To have a constructive model is conceptually important; constructive algebraic geometry?

This research can play a rôle when internalizing these models, e.g. importance of having a *tiny* interval to get a universe for directed univalence.

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