Are Universes Open or Closed?

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Types and Topology
A workshop in honour of
Martín Escardó's 60th birthday,
Birmingham, 17-18 December 2025

A conversation

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Universes are open

- Martín's **Type Topology** is implemented in Agda. It has a tower of universes

$$\mathrm{Set}_0, \mathrm{Set}_1, \mathrm{Set}_2, \ldots$$

- When you start a new development from scratch, Set_l are all empty, except that the type of universe levels

$$\mathrm{Level} : \mathrm{Set}_0$$

  You add new elements to Set_l by adding new **data** types. These are inductively (or inductive-recursively or inductive-inductively) defined.

- (Recent option `-level-universe` makes Level live in its own universe $\mathrm{LevelUniv}$.)

Universes are closed

- Universes à la Tarski are special kinds of inductive-recursive definitions.
- Inductive-recursive definitions have elimination rules.

Two important questions

- What are good rules for universes in a type theory system, e g the one underlying Agda?
- Constructive validity of universes?

Martin-Löf type theory 1986

A "logical framework" with distinction between "set" and "type":

$$\text{Set type} \qquad \frac{A : \text{Set}}{A \text{ type}} \qquad \frac{\alpha \text{ type} \quad \begin{matrix}(x:\alpha)\\ \vdots \\ \beta \text{ type}\end{matrix}}{(x:\alpha) \to \beta \text{ type}}$$

The idea:

1973,1979	1986
$\mathbb{N}$ type	$\mathbb{N} : \text{Set}$
$\dfrac{a : \mathbb{N}}{\mathrm{s}(a) : \mathbb{N}}$	$\mathrm{s} : \mathbb{N} \to \mathbb{N}$

Sets are closed and types are open

Martin-Löf's 1979 notion of constructive validity (meaning explanations) is extended to account for the 1986 type-set distinction:

Sets A is a set provided you know *how to form its elements* and *when two elements are equal*. Sets are closed (inductively or inductive-recursively generated). For example we have a tower of Tarski-universes:

$$\begin{aligned} \mathrm{U}_n &: \mathrm{Set} \\ \mathrm{T}_n &: \mathrm{U}_n \to \mathrm{Set} \end{aligned}$$

Types α is a type provided you know *what its objects are* and *when two objects are equal*. Types can be open. No question of adding an elimination rule for the type Set.

Martin-Löf 1986 and Agda (ALF 1991)

Sets are closed and types are open:

Martin-Löf	Agda	
	$\vdots$	
	Set_2	
	Set_1	open (Russell)
Set	Set_0	
$\vdots$	$\vdots$	
U_2	(U_2)	
U_1	(U_1)	closed, IR (Tarski)
U_0	(U_0)	

Agda has universe polymorphism:

$$\alpha : \mathrm{Level} \vdash \mathrm{Set}_\alpha$$

Martin-Löf 1986 and Agda

Martin-Löf	Agda	
	$\vdots$	
	(Set_ω)	
	$\vdots$	
	Set_2	
	Set_1	open (Russell)
Set	Set_0	
$\vdots$	$\vdots$	
U_2	(U_2)	
U_1	(U_1)	closed, IR (Tarski)
U_0	(U_0)	

Martín's notation in Type Topology

Agda	Type Topology
$l : \mathrm{Level}$	$\mathcal{U} : \mathrm{Universe}$
$A : \mathrm{Set}_l$	$A : \mathcal{U}\,^{\cdot}$

Reasons?

- "Set" suggests "h-set" in HoTT/UF.
- "Level" suggests natural number.
- "Universe" suggests a more general notion.

BCDE – Universe level as judgment

- BCDE = Bezem, Coquand, D, Escardó 2022: *Type Theory with Explicit Universe Polymorphism*
- cf Courant 2002

New judgment forms (Type Topology notation):

$$\mathrm{U} : \mathrm{Universe} \qquad\qquad \mathrm{U} = \mathrm{V}$$

Operations:

$$\mathcal{U} : \mathrm{Universe} \quad \vdash \quad \mathcal{U}^+ : \mathrm{Universe}$$
$$\mathcal{U}, \mathcal{V} : \mathrm{Universe} \quad \vdash \quad \mathcal{U} \vee \mathcal{V} : \mathrm{Universe}$$

Note: $\mathcal{U}, \mathcal{V}$ universe variables.

BCDE – Universe level as judgment

Some rules:

$$\mathcal{U} : \text{Universe} \quad \vdash \quad \mathcal{U}^{\cdot} \text{ type}$$
$$\mathcal{U} : \text{Universe} \quad \vdash \quad \mathbb{N}^{\mathcal{U}} : \mathcal{U}^{\cdot}$$

Moreover:

- No first universe $\text{U}_0 : \text{Universe}$.
- Universe quantification $[\mathcal{U}]A$ type.
- Equational universe constraints (cf Voevodsky).
- (BCDE-paper is Tarski-style while $\mathcal{U}^{\cdot}$ in Type Topology (Agda) is Russell-style.)

What is the abstract/general notion of universe?

- All inductive-recursive definitions can be seen as Tarski-style universes in a generalized sense. They reflect certain operations.
- So, abstractly, universes and inductive-recursive definitions are maybe the same?

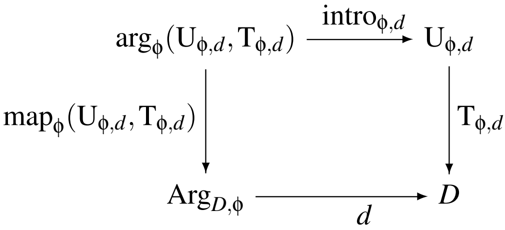
Universe à la Tarski as reflection principle

Closure under Σ:

$$
\begin{array}{ccc}
(a:\mathrm{U})\times(\mathrm{T}(a)\to\mathrm{U}) & \xrightarrow{\hat{\Sigma}} & \mathrm{U} \\
{\scriptstyle (a,b)\mapsto(\mathrm{T}(a),\mathrm{T}\circ b)}\downarrow & & \downarrow{\scriptstyle \mathrm{T}} \\
(A:\mathrm{Set})\times(A\to\mathrm{Set}) & \xrightarrow[\Sigma]{} & \mathrm{Set}
\end{array}
$$

We work in (extension of) Martin-Löf LF-version of type theory 1986.

Induction-recursion as general reflection principle



- D is a type
- $\phi : \mathrm{SP}_D$ is a code for "strict positivity" ensuring that $\mathrm{Arg}_{D,\phi}$ strictly positive in D.
- for any d, as above, we can inductive-recursively generate $\mathrm{U}_{\phi,d}, \mathrm{T}_{\phi,d}$.
- See D, Setzer, TLCA 1999 for definition of $\mathrm{Arg}, \mathrm{arg}, \mathrm{map}$ and elimination rule.

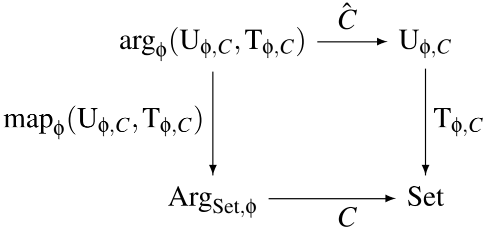
Universes as special cases of induction-recursion

Induction-recursion states that we can reflect an arbitary function

$$d : \mathrm{Arg}_{D,\phi} \to D$$

into an arbitrary type D. A natural restriction for universe-like induction-recursion is

- $D = \mathrm{Set}$
- $d = C$ encodes (sequences of) set formers (canonical) such as $\Pi, \Sigma, +, \mathbb{N}, \ldots, \mathrm{U}_0, \ldots$.



Mahlo principle in type theory (idea)

For any f, as below, there is a universe $(\mathrm{U}_f, \mathrm{T}_f)$ closed under f:

$$
\begin{array}{ccc}
\mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) & \xrightarrow{\hat{f}} & \mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) \\
\Big\downarrow {\scriptstyle \mathrm{T}_f^{\mathrm{Fam}}} & & \Big\downarrow {\scriptstyle \mathrm{T}_f^{\mathrm{Fam}}} \\
\mathrm{Fam}(\mathrm{Set}) & \xrightarrow[f]{} & \mathrm{Fam}(\mathrm{Set})
\end{array}
$$

- $\mathrm{Fam}(\mathrm{Set}) = (A : \mathrm{Set}) \times (B \to \mathrm{Set})$
- $\mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) = (a : \mathrm{U}_f) \times (\mathrm{T}_f(a) \to \mathrm{U}_f)$
- $\mathrm{T}_f^{\mathrm{Fam}}(a, b) = (\mathrm{T}_f(a), \mathrm{T}_f \circ b)$
- If f is Palmgren's next universe operator, then $(\mathrm{U}_f, \mathrm{T}_f)$ is Palmgren's super universe.

Mahlo principle in type theory

- Set becomes a Mahlo universe in the sense of Setzer 2000.
- $(\mathrm{U}_f, \mathrm{T}_f)$ are subuniverses of Set and instances of inductive-recursive definitions in the sense of D, Setzer 1999.
- However, they are not necessarily universes in the sense discussed before, since they can reflect arbitrary functions f, not only set constructors C.

Constructive validity in Martin-Löf 1979

- A type is valid provided
  - $A \Rightarrow C(a_1, \ldots, a_m)$ where C is a type constructor,
  - premisses of formation rule express requirements on $a_1, \ldots, a_m$.
- $a : A$ is valid provided
  - $A \Rightarrow C(a_1, \ldots, a_m)$ where C is a type constructor,
  - $a \Rightarrow c(b_1, \ldots, b_n)$, where c is a term constructor for C
  - premisses of introduction rules express requirements on $a_1, \ldots, a_m, b_1, \ldots, b_n$.

Checking the validity of a judgment leads to a repeated process of lazy evaluation and constructor matching. This process has to be well-founded and can be visualized as a well-founded tree.

Constructive validity of universes

- A type is valid provided
  - $A \Rightarrow \mathrm{U}_0$,
  - or $A \Rightarrow \mathbb{N}$,
  - or $A \Rightarrow B + C$ provided B, C type,
  - etc, for the remaining type formers
- $a : \mathrm{U}$ (Russell) is valid provided
  - $a \Rightarrow \mathbb{N}$,
  - or $a \Rightarrow b + c$ provided $b, c : \mathrm{U}$,
  - etc, for the remaining small type formers

Constructive validity of $\mathcal{S}et$ (Mahlo)?

Martin-Löf extended his 1979 meaning explanations to the 1986 type theory with the “open” type $\mathcal{S}et$. Why is

$$\mathrm{U}_f : \mathrm{Set}$$

for any

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})?$$

It seems as though we have a good construction principle, f is a "parameter"... generalizes superuniverse, supersuperuniverse, etc.

Constructive validity of Set (Mahlo)?

Not clear enough ... we should extend Martin-Löf 1979 so that also the type Set is "closed". However, just using the previous pattern does not work: $A : \mathrm{Set}$ (Russell-style Mahlo) is valid provided

- $A \Rightarrow \mathbb{N}$,
- or $A \Rightarrow B + C$ provided $B, C : \mathrm{Set}$,
- etc for the other set formers,
- or $A \Rightarrow \mathrm{U}_f$ provided $f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$.

But this leads to a non-well-founded process of lazy evaluation and constructor matching!

Constructive validity of the Mahlo universe

- Instead we have $A : \mathrm{Set}$ if $A \Rightarrow \mathrm{U}_f$ and

$$f \circ \mathrm{T}_f^{\mathrm{Fam}} : \mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) \to \mathrm{Fam}(\mathrm{Set})$$

  the diagonal in

$$\begin{array}{ccc}
\mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) & \xrightarrow{\hat{f}} & \mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) \\
{\scriptstyle \mathrm{T}_f^{\mathrm{Fam}}}\downarrow & \searrow & \downarrow{\scriptstyle \mathrm{T}_f^{\mathrm{Fam}}} \\
\mathrm{Fam}(\mathrm{Set}) & \xrightarrow[f]{} & \mathrm{Fam}(\mathrm{Set})
\end{array}$$

- Recall subuniverse formation: $\mathrm{U}_f : \mathrm{Set}$ for any

$$f : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Fam}(\mathrm{Set})$$

“Predicative” model of the Mahlo universe in set theory

D, Setzer 2025: *Predicativity of the Mahlo Universe in Type Theory*.

- Inductive-recursively generate the subuniverses $(\mathrm{U}_f, \mathrm{T}_f)$ for all f (not only well-typed).
- Inductively generate Set, with clauses

$$\frac{f \circ \mathrm{T}_f^{\mathrm{Fam}} : \mathrm{Fam}(\mathrm{U}_f, \mathrm{T}_f) \to \mathrm{Fam}(\mathrm{Set})}{\mathrm{U}_f \in \mathrm{Set}}$$

$$\frac{}{\mathbb{N} \in \mathrm{Set}}$$

$$\vdots$$

Two answers

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Happy birthday, Martín!