The "Collatz Problem" of Domain Theory: Is **FS** = **RB**?

Types and Topology: A Workshop in honor of Martín Escardó's 60th birthday

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Outline

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A personal prologue: Martín's 5% rule

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- His advice (roughly):
  - Spend *95%* of your time finishing your thesis.
  - Spend at most *5%* of your time playing with this problem.

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FS = **RB** ?

- His advice (roughly):
  - Spend *95%* of your time finishing your thesis.
  - Spend at most *5%* of your time playing with this problem.
- This talk is my belated 5%:
  now that I have earned my tenure, I can afford to 'waste' time *intelligently* on this Collatz–like puzzle in domain theory.

Why a "Collatz problem" for domain theory?

- Like the Collatz conjecture, a.k.a. $3n + 1$-problem[1]:
  - Very easy to state.
  - No consensus that current tools are strong enough.
  - Dangerous to one's research time!

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- For **FS** = **RB**?:
  1. Domain theory may simply *not be ready* yet.
  2. Anyone who devotes serious time to it risks *wasting* that time.
  3. Yet the question is *fundamental*: it compares two core finitary models of continuity in denotational semantics.

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  3. Yet the question is *fundamental*: it compares two core finitary models of continuity in denotational semantics.
- I will:
  - Recall the key players (**FS**, **RB**, Lawson's disc domain),
  - Explain why Lawson's disc domain is FS with some new insight,
  - And indicate how far we are from resolving **FS** = **RB**.

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From posets to domains

Definition (Dcpo)

A poset $(P, \leq)$ is *directed complete* if every directed subset has a supremum.
A dcpo which has the least element $\perp$ is a pointed dcpo.[2]

- directed: $D \neq \varnothing$ and $\forall d_1, d_2 \in D.\ \exists d_3 \in D.\ d_1, d_2 \leq d_3$.
- supremum: least upper bound of D (if it exists).

[2] In this talk, all dcpo's are pointed.

The dcpo of closed discs

Let

$$L := \left\{ D(a,r) := \{x \in \mathbb{R}^2 : d(x,a) \le r\} \mid a \in \mathbb{R}^2, r \ge 0 \right\} \cup \{\mathbb{R}^2\}$$

be the collection of all *closed discs* in $\mathbb{R}^2$, including degenerate ones $\{a\}$ and the super-huge one $\mathbb{R}^2$.

Order L by reverse inclusion:

- $\mathbb{R}^2 \supseteq D$ for all $D \in L$ (so $\mathbb{R}^2$ is bottom);
- $D(a,r) \le D(b,s)$ iff $D(a,r) \supseteq D(b,s)$.

Then $(L, \le)$ is a pointed dcpo: directed suprema are given by filtered intersections of discs.

The dcpo of closed discs (geometry)

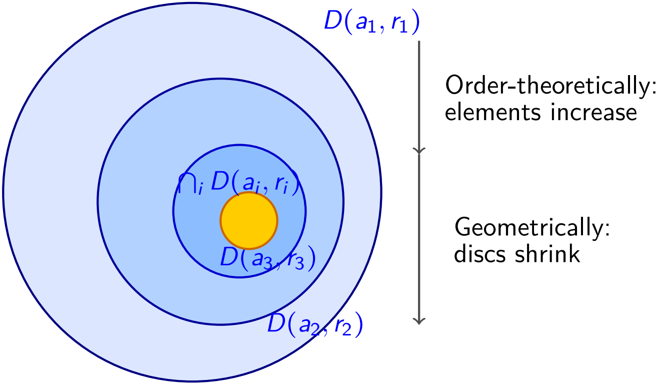

Figure 1: The disc dcpo $(L, \supseteq)$: $D(a, r) \leq D(b, s) \iff \|a - b\| + s \leq r$.

Domains and the way-below relation

Way-below relation

For a dcpo P:

$$y \ll x \iff \forall D \subseteq_{\text{dir}} P.\ \left(\bigvee{}^{\uparrow} D \geq x \Longrightarrow \exists d \in D.\ d \geq y\right).$$

Domains and the way-below relation

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Definition (Domain = Continuous dcpo)

A subset $B \subseteq P$ is a *basis* of P if for every $x \in P$,

$$B_x := \downarrow\!\!\!\downarrow x \cap B$$

contains a directed subset with supremum x.

A dcpo is called a *continuous dcpo* or *domain* if it has such a basis.

The Lawson's disc domain

In $(L, \leq)$,

$$D(a,r) \ll D(b,s) \iff \mathrm{int}(D(a,r)) \supseteq D(b,s),$$

and $\mathbb{R}^2 \ll D$ for all $D \in L$.
Indeed, for any $D \in L$,

$$D = \bigcap\nolimits^{\downarrow} \{D' \in L \mid D' \ll D\},$$

which shows that $(L, \leq)$ is a pointed domain, called the *Lawson's disc domain*.

Fact: $(L, \leq)$ is not algebraic: it has not enough compact elements.

Approximate identities

Definition (Approximate identity)

Let P be a dcpo. An *approximate identity* on P is a directed subset $\mathcal{D} = (\delta_i)_{i \in I}$ of the function space $[P \to P]$ of all Scott-continuous self-maps on P, ordered pointwise, such that

$$\bigvee_{i \in I}^{\uparrow} \delta_i = \mathrm{id}_P,$$

i.e., the pointwise supremum of all $\delta_i \in \mathcal{D}$ is the identity map on P.

FS domains: finitely separating maps

Definition (Finitely separating)

Let P be a dcpo. A Scott-continuous function $\delta : P \to P$ is *finitely separating* if there exists a finite set $F_\delta \subseteq P$ such that

$$\forall x \in P.\ \exists y \in F_\delta.\ \delta(x) \leq y \leq x.$$

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A dcpo P is *finitely separated* if it has an approximate identity $\mathcal{D} \subseteq [P \to P]$ consisting entirely of finitely separating maps.

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Theorem

Every finitely separated dcpo is continuous, henceforth called an FS domain.

RB and FS in the domain zoo

$$\underbrace{\mathbf{B} \subseteq \mathbf{RB} \subseteq \overbrace{\mathbf{FS}}^{\text{maximal}}}_{\text{CCC subcategories of } \mathbf{Dom}} \subseteq \mathbf{Dom}$$

Class	Key Property	Intuitive Idea
Bifinite	Has an approximate identity of idempotent deflations (δ_i) with finite image	$\mathrm{K}(P) = \bigcup_i \delta_i[P]$.
RB	Has an approximate identity of deflations (δ_i) with finite image	Retracts of bifinite domains (δ_i not necessarily idempotent)
FS	Has an approximate identity of finitely separating maps δ	Separation by a finite set F_δ
Domain	$\ll$ is approximating	Directed approximations

AlgFS = B and Dom = RAlg

- **Fact**: every continuous domain is a Scott-continuous retract of an algebraic domain:

Dom = **RAlg**.

AlgFS = B and Dom = RAlg

- **Fact**: every continuous domain is a Scott-continuous retract of an algebraic domain:

$$\mathbf{Dom} = \mathbf{RAlg}.$$

- Gierz et al. (Prop. II-2.20):

Theorem

For a dcpo P, the following are equivalent:

1. *P is an algebraic FS domain.*
2. *P is an algebraic domain and has an approximate identity consisting of maps with finite image.*
3. *P has an approximate identity of idempotent deflations with finite image, i.e., P is bifinite.*

So

$$\mathbf{B} = \mathbf{AlgFS}.$$

The open problem (35 years!)

Because

$$\boxed{\mathbf{Dom} = \mathbf{RAlg}} \quad \text{and} \quad \boxed{\mathbf{B} = \mathbf{AlgFS}},$$

one asks:

Open Problem (ca. 35 years)

$$\text{Is } \boxed{\mathbf{FS} = \mathbf{RAlgFS} = \mathbf{RB}}\ ?$$

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$$\text{Is } \boxed{\mathbf{FS} = \mathbf{RAlgFS} = \mathbf{RB}} \text{ ?}$$

- No counterexample known.
- No general proof known.
- Lawson's disc domain is a natural test case.

Lawson's disc domain as a test case

In Section 4.2.2 of [2], Achim Jung concluded with a concrete example of an FS-domain, due to Jimmie Lawson ([5]).

Lawson's disc domain as a test case

In Section 4.2.2 of [2], Achim Jung concluded with a concrete example of an FS-domain, due to Jimmie Lawson ([5]).
It was (and still is) a *likely candidate* for an FS-domain that is not RB.

It is none other than the Lawson disc domain L.

Lawson's approximate identity on L

Let L denote the Lawson disc domain.

For each $\varepsilon > 0$, define $f_\varepsilon : L \to L$ by

$$f_\varepsilon(D) = \begin{cases} D(a, r + \varepsilon) & \text{if } D = D(a,r) \subseteq B\left(0, \frac{1}{\varepsilon}\right); \\ \mathbb{R}^2 & \text{otherwise.} \end{cases}$$

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Interpretation: for small discs strictly inside $B(0, 1/\varepsilon)$, we *inflate* the radius by ε; for all others we collapse to $\mathbb{R}^2$ (the bottom of L).

Why naive definition will not give FS

Consider the functions of the form

$$g_\varepsilon(D) = \begin{cases} D(a,r) & \text{if } D = D(a,r) \subseteq B\left(0, \frac{1}{\varepsilon}\right); \\ \mathbb{R}^2 & \text{otherwise.} \end{cases}$$

(without inflation).

- They also form an approximating identity on L.
- But they **fail** to be finitely separating:
  - We would need a finite separating set F_ϵ working for uncountably many pairs $D(a,r) \leq D(a,r)$.
  - This is impossible.

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  - This is impossible.
- Moral: to get finite separation, we must use a *robust gap*:

$$D(a,r) \subseteq m_D \subseteq D(a, r+\varepsilon)$$

that persists under small perturbations of the disc.

Robustness via S_ε

Demarcate the region where f_ε inflates discs:

$$S_\varepsilon := \{B(c, r) : \|c\| + r < 1/\varepsilon\}.$$

Thus, S_ε consists of discs lying strictly inside the open ball $B\left(0, \frac{1}{\varepsilon}\right)$.

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Lemma (Uniform Robustness)

Fix $\varepsilon > 0$. *For each* $D_p = B(c_0, r_0) \in S_\varepsilon$ *define*

$$m_{D_p} := B\left(c_0,\ r_0 + \frac{\varepsilon}{2}\right).$$

Then with $\delta_\varepsilon := \varepsilon/4$ *the following holds:*
For any $D_p, D_q \in S_\varepsilon$ *with parameter triples* $p = (c_0, r_0)$ *and* $q = (c, r)$ *satisfying* $\|q - p\| < \delta_\varepsilon$ *in* $\mathbb{R}^3$, *we have*

$$f_\varepsilon(D_q) \le m_{D_p} \le D_q.$$

Uniform robustness: picture

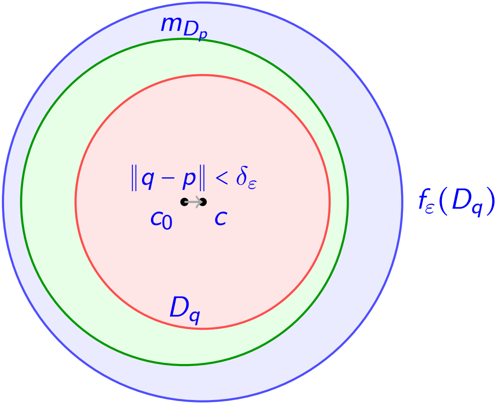

Figure 2: Uniform robustness: $D_q \subseteq m_{D_p} \subseteq f_\varepsilon(D_q)$.

Total boundedness of the parameter set

The parameter set of discs in S_ε is

$$K_\varepsilon := \{(c, r) : \|c\| + r < 1/\varepsilon\},$$

which lies in the compact box

$$T_\varepsilon := \{(c, r) : \|c\| \le 1/\varepsilon,\ 0 \le r \le 1/\varepsilon\} \subseteq \mathbb{R}^3.$$

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Thus K_ε is totally bounded; for $\delta_\varepsilon = \varepsilon/4$ there exist finitely many points

$$p_1, \dots, p_k \in K_\varepsilon$$

such that every $q \in K_\varepsilon$ is within δ_ε of some p_j.

Finite separation using robustness + total boundedness

For each j define $m_j := m_{D_{p_j}} = B\left(c_j, r_j + \frac{\varepsilon}{2}\right)$.

Proposition

Let $M_\varepsilon := \{\mathbb{R}^2, m_1, \ldots, m_k\}$. *Then for every* $D \in L$ *there exists* $m \in M_\varepsilon$ *with*

$$f_\varepsilon(D) \leq m \leq D.$$

- If $D \notin S_\varepsilon$ then $f_\varepsilon(D) = \mathbb{R}^2$ and $m = \mathbb{R}^2$ works.
- If $D \in S_\varepsilon$ then its parameter $q \in K_\varepsilon$ is within δ_ε of some p_j, and Uniform Robustness yields $f_\varepsilon(D) \leq m_j \leq D$.

Hence each f_ε is finitely separated.

What remains annoyingly open

- We have seen:
  - Lawson's disc domain L is a continuous domain.
  - The Lawson maps f_ε form an approximate identity.
  - Each f_ε is finitely separated (via robustness + total boundedness).
  - Hence L is an FS-domain.

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- We have seen:
  - Lawson's disc domain L is a continuous domain.
  - The Lawson maps f_ε form an approximate identity.
  - Each f_ε is finitely separated (via robustness + total boundedness).
  - Hence L is an FS-domain.
- It is still not known whether L is an RB-domain.
- If L were *not* RB, it would settle **FS** $\neq$ **RB**.
- If L were RB, we would need to search for another "Collatz–like" example.

What we already know (Zou–Li–Guo, 2018)

- Each finitely separating $\delta : P \to P$ can, by the Axiom of Choice, be represented by a function $f_\delta : P \to P$ with *finite image* such that

$$\forall x \in P.\ \delta(x) \leq f_\delta(x) \leq x.$$

- If f_δ is additionally *order-preserving*, then δ is called **super finitely separating (SFS)**.
- **Theorem.** A dcpo P is an **RB-domain** iff it admits an approximate identity consisting of SFS maps.

The problem FS = RB reduces to bridging FS and SFS.

The current frontier of the FS = RB problem

- After Zou–Li–Guo, the problem is purely **order-theoretic**:

$$\mathbf{FS} = \mathbf{RB}$$

if and only if

every f.s. map admits a monotone finite-image refinement.

- Known coincidences:

Domain class	FS	RB coincide?
Algebraic	✓	✓
L-domain	✓	✓
(dual) Consistent join-semilattice	✓	✓
General continuous dcpo	✓	?

- In general, no known way to enforce monotonicity without losing finiteness or continuity.

Possible lines of attack

1. Prove or disprove that the Lawson approximating identity $(f_\varepsilon)_{\varepsilon>0}$ can admit a monotone finite-image refinement.
2. Suppose $(L, \leq)$ is an RB with an approximating identity comprising SFS maps. Relate these with the existing $(f_\varepsilon)_{\varepsilon>0}$ and get a contradiction, possibly a geometric one.
3. Develop a completely new set of tools.

Closing thoughts: a Collatz-style open problem

- We now understand **FS = RB** ε-better than 35 years ago:
  - FS-ness of L can be justified through a certain robustness condition.
  - RB is characterised by *super* finitely separating maps.
  - We know many positive coincidence results (algebraic, L-domains, etc.).

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- But the core question

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  - We know many positive coincidence results (algebraic, L-domains, etc.).
- But the core question

  FS = **RB** ?

  remains stubbornly open.
- Perhaps domain theory is still not ready.
  Perhaps we need new tools.
  In the meantime, this problem serves as our Collatz: a warning label on how we spend our mathematical time.

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Z. Zou, Q. Li, and L. Guo, A note on the problem when FS-domains coincide with RB-domains, *Categories and General Algebraic Structures with Applications*, 8(1), 51–59, 2018.

Algebraic domains (backup)

Definition (Compact elements)

An element x of a dcpo P is *compact* if $x \ll x$.
The collection of all compact elements of P is denoted by $\mathsf{K}(P)$.

Definition (Algebraic domains)

A dcpo P is an *algebraic domain* if it has $\mathsf{K}(P)$ as a basis.

Example & Non-example

1. The powerset lattice $(\mathbb{P}(X), \subseteq)$ is algebraic.
2. The Lawson's disc domain $(L, \supseteq)$ is **not** algebraic.

Continuous = R-Algebraic (backup)

Definition (Scott-continuous retraction)

Let P and Q be dcpo's, and Scott-continuous mappings between them

$$P \underset{r}{\overset{s}{\rightleftarrows}} Q$$

such that $r \circ s = \mathrm{id}_P$.
We call $r : Q \longrightarrow P$ a *Scott-continuous retraction*, and P is a (Scott-continuous) *retract* of Q.

Theorem

Every continuous domain is the retract of an algebraic domain, i.e.,

$$\mathbf{Dom} = \mathbf{RAlg}.$$

Flat natural numbers is bifinite (backup)

The sequence, $(\delta_k(N))_{k\in\mathbb{N}}$, of finite posets has bilimit $N = \mathbb{N}_\perp$.

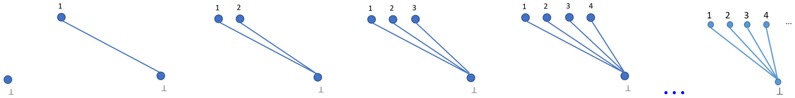

Figure 3: N is bifinite.

A simple FS example on $[0,1]$ (backup)

Consider the unit interval $[0,1]$ with the usual order $\leq$.
For each $\epsilon > 0$, the function

$$f_\epsilon(x) := \max\{x - \epsilon, 0\}$$

is finitely separated from the identity map $\mathrm{id}_{[0,1]}$.
Of course, $([0,1], \leq)$ being a bc-domain must be FS.

Proof of Uniform Robustness (backup)

Proof.

As usual, $B(c_1, r_1) \subseteq B(c_2, r_2)$ iff $\|c_1 - c_2\| + r_1 \leq r_2$.
If $\|q - p\| < \delta_\varepsilon$ then $\|c - c_0\| < \delta_\varepsilon$ and $|r - r_0| < \delta_\varepsilon$. Thus

$$\|c - c_0\| + r \leq r_0 + 2\delta_\varepsilon = r_0 + 2 \cdot \frac{\varepsilon}{4} = r_0 + \frac{\varepsilon}{2}$$

showing $D_q \subseteq m_{D_p}$.
Since $D_q \in S_\varepsilon$, we have $f_\varepsilon(D_q) = B(c, r + \varepsilon)$. Also

$$\|c - c_0\| + \left(r_0 + \frac{\varepsilon}{2}\right) \leq \delta_\varepsilon + r_0 + \frac{\varepsilon}{2} \leq \left(r + \frac{\varepsilon}{4}\right) + \frac{3\varepsilon}{4} = r + \varepsilon,$$

establishing $m_{D_p} \subseteq f_\varepsilon(D_q)$. □