A constructive theory of uniformity and its application to integration theory

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Motivations

Motivations

Errett Bishop, the founder of neutral constructive mathematics (or Bishop's constructive mathematics), wrote in his book as follows [1].

> *The situation is easily summarized: Nonmetric spaces and nonseparable metric spaces play no significant role in those parts of analysis with which this book is concerned. To illustrate this point, consider the concept of a uniform space, as developed in Probs. 17 to 21 of Chap. 4. A uniform space at first sight appears to be a natural and fruitful concept for constructive mathematics, a promising substitute for the concept of a topological space. In fact, this is not the case.*

[1] Bishop 1967, Appendix A: Metrizability and Separability; he introduced the notion of a uniform space using a family of pseudometrics.

Motivations

- Although classically,
  - the topology on the space l_∞ of bounded sequences of real numbers is given by the norm

$$\|(x_n)_{n\in\mathbb{N}}\| = \sup\{|x_n| \mid n \in \mathbb{N}\};$$

  - the strong topology on the dual space (the set of bounded linear functionals) E^* of a normed space E is given by the norm

$$\|f\| = \sup\{|f(x)| \mid x \in E, \|x\| \leq 1\},$$

- constructively, they are uniform topologies but are not given by any family of pseudometrics.
- We need a more general framework than a family of pseudometrics for defining the notion of a uniform space.

Motivations

Here, as an application of a general framework for uniform spaces, we consider integration theory.

- One of the motivations Lebesgue developed his integration theory was to make integration and limit commute:

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n,$$

  which does not hold for the Riemann integral.
- The Lebesgue integral is based on the Lebesgue measure which is a generalisation of the notions of a length, an area and a volume.

Motivations

- Since a measure is defined on a σ-algebra which is closed under the complementation,
- the lack of law of excluded middle in constructive mathematics brings us a difficulty to define an appropriate domain of a measure.
- Bishop overcame the difficulty by introducing the notion of a complemented set, and developed a constructive measure and integration theory.
- However, the original motivation of Lebesgue is concerned with the topological notion of a limit.

Motivations

- As far as we are concerned with convergence theorems such as the monotone and dominated convergence theorems of Lebesgue,
- we may be able to constructively deal with them topologically without invoking the notion of a measure and the notion of a complemented set.
- Spitters [Spitters 2006] took an approach using Bishop's notion of a uniform space, and following Bishop's advice [Bishop 1967, Preface].

Preliminaries

Constructive set theory

The elementary constructive (and predicative) set theory ECST was introduced by Aczel and Rathjen.

The axioms and rules of ECST are those of intuitionistic predicate logic with equality. In addition, ECST has the set theoretic axioms

Extensionality: $\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b]$;

Pairing: $\forall a \forall b \exists c \forall x (x \in c \leftrightarrow x = a \vee x = b)$;

Emptyset: $\exists a \forall x (x \notin a)$;

Union: $\forall a \exists b \forall x [x \in b \leftrightarrow \exists y \in a\, (x \in y)]$;

Replacement:

$$\forall a [\forall x \in a \exists ! y \varphi(x, y) \rightarrow \exists b \forall y (y \in b \leftrightarrow \exists x \in a\, \varphi(x, y))]$$

for all formulae $\varphi(x, y)$, where b is not free in $\varphi(x, y)$;

Constructive set theory

Bounded Separation:

$$\forall a \exists b \forall x (x \in b \leftrightarrow x \in a \wedge \varphi(x))$$

for all bounded formulae $\varphi(x)$, where b is not free in $\varphi(x)$; here a formula $\varphi(x)$ is bounded, or Δ_0, if all its quantifiers are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$;

Strong Infinity:

$$\exists a[0 \in a \wedge \forall x(x \in a \rightarrow x+1 \in a) \\ \wedge \forall y(0 \in y \wedge \forall x(x \in y \rightarrow x+1 \in y) \rightarrow a \subseteq y)].$$

Constructive set theory

In ECST, we are able to perform basic set constructions in mathematical practice.

- the ordered pair $(x, y) = \{\{x\}, \{x, y\}\}$ of x and y, using Pairing;
- the *cartesian product* $A \times B$ of sets A and B consisting of the ordered pairs (x, y) with $x \in A$ and $y \in B$, using Replacement and Union.

In addition to the axioms of ECST, we assume Exponentiation Axiom asserting that the class B^A of functions from a set A into a set B, called the exponential of A and B, forms a set.

Exponentiation: $\forall a \forall b \exists c \forall f (f \in c \leftrightarrow f \in b^a)$.

Constructive set theory

We write hom(I, I') for the set of monotone functions between preordered sets $I = (\underline{I}, \preceq_I)$ and $I' = (\underline{I'}, \preceq_{I'})$.

A preordered set $I = (\underline{I}, \preccurlyeq_I)$ is directed if

- $\underline{I}$ is inhabited;
- there exists $\mathrm{ub}_I \in \mathrm{hom}(I \times I, I)$, called an upper bound function, such that for each $x, y \in \underline{I}$, $x \preccurlyeq_I \mathrm{ub}_I(x, y)$ and $y \preccurlyeq_I \mathrm{ub}_I(x, y)$.

A setoid (or Bishop set) X is a pair $(\underline{X}, =_X)$ of a set $\underline{X}$ and an equivalence relation $=_X$ on $\underline{X}$.

Setoids

Let $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_Y)$ be setoids. Then

- a function $f : \underline{X} \to \underline{Y}$ is a setoid mapping of X into Y if

$$x =_X y \Rightarrow f(x) =_Y f(y)$$

  for all $x, y \in \underline{X}$; we then write $f : X \to Y$.

- Two setoid mappings $f, g : X \to Y$ are identical, denoted by $f \sim g$, if

$$x =_X y \Rightarrow f(x) =_Y g(y)$$

  for all $x, y \in \underline{X}$, or equivalently $f(x) =_Y g(x)$ for all $x \in \underline{X}$.

- A setoid mapping $f : X \to Y$ is a setoid injection if

$$f(x) =_Y f(y) \Rightarrow x =_X y$$

  for all $x, y \in \underline{X}$.

Setoids

The product $X \times Y$ of setoids $X = (\underline{X}, =_X)$ and $Y = (\underline{Y}, =_X)$ is a pair of the set $\underline{X} \times \underline{Y}$ and an equivalence relation $=_{X \times Y}$ on $\underline{X} \times \underline{Y}$ given by

$$(x, y) =_{X \times Y} (x', y') \Leftrightarrow x =_X x' \text{ and } y =_Y y'$$

for all $(x, y), (x', y') \in \underline{X} \times \underline{Y}$; in which case, the projections are setoid mappings.

Uniform spaces

Fundamental definitions

A uniform structure on a setoid $X = (\underline{X}, =_X)$ is a triple consisting of a directed preordered set $I_X = (\underline{I}_X, \preccurlyeq_{I_X})$, a function $\rho_X \in \hom(I_X, I_X)$, and a relation $\Vdash_X$ between $\underline{X} \times \underline{X}$ and $\underline{I}_X$ such that

1. for all $x, y \in \underline{X}$, $x =_X y$ if and only if $(x, y) \Vdash_X a$ for all $a \in \underline{I}_X$;
2. for all $a \in \underline{I}_X$ and $x, y, x', y' \in \underline{X}$, if $x =_X x'$, $y =_X y'$ and $(x, y) \Vdash_X a$, then $(x', y') \Vdash_X a$;
3. for all $a \in \underline{I}_X$ and $x, y \in \underline{X}$, if $(x, y) \Vdash_X a$, then $(y, x) \Vdash_X a$;
4. for all $a, b \in \underline{I}_X$ and $x, y \in \underline{X}$, if $a \preccurlyeq_{I_X} b$ and $(x, y) \Vdash_X b$, then $(x, y) \Vdash_X a$;
5. for all $a \in \underline{I}_X$ and $x, y, z \in \underline{X}$, if $(x, y) \Vdash_X \rho_X(a)$ and $(y, z) \Vdash_X \rho_X(a)$, then $(x, z) \Vdash_X a$.

A uniform space is a setoid equipped with a uniform structure.

Fundamental definitions

Example 1

Let $\underline{X}$ be a set, and let $d : \underline{X} \times \underline{X} \to \mathbb{R}$ be a pseudometric on $\underline{X}$. Then a binary relation $=_X$ on $\underline{X}$, given by

$$x =_X y \Leftrightarrow d(x, y) = 0$$

for all $x, y \in \underline{X}$, is an equivalence relation; hence $X = (\underline{X}, =_X)$ is a setoid.
Let ρ_X and $\Vdash_X$ be a monotone function from $\mathbb{N}$ into $\mathbb{N}$ and a relation between $\underline{X} \times \underline{X}$ and $\mathbb{N}$ defined by

$$\rho_X(n) = n + 1,$$
$$(x, y) \Vdash_X n \Leftrightarrow d(x, y) \leq 2^{-n}$$

for all $n \in \mathbb{N}$ and $x, y \in \underline{X}$, respectively. Then $(\mathbb{N}, \rho_X, \Vdash_X)$ is a uniform structure on the setoid X.

Fundamental definitions

Let X be a setoid, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Then a function $\boldsymbol{x} : j \mapsto x_j$ from $\underline{J}$ into $\underline{X}$ is called a net (or Moore-Smith sequence) in X on J, and is denoted by $(x_j)_{j \in \underline{J}}$; a net $(x_n)_{n \in \mathbb{N}}$ on the linearly ordered set $(\mathbb{N}, \leq)$ is called a sequence in X; we write $\underline{X}^J$ for the set $\underline{X}^{\underline{J}}$ of nets in X on J.

Let $(I_X, \rho_X, \Vdash_X)$ be a uniform structure on X. Then a net $\boldsymbol{x} = (x_j)_{j \in \underline{J}} \in \underline{X}^J$ converges to an element x of $\underline{X}$ in X if there exists $\beta \in \hom(I_X, J)$, called a modulus (of convergence), such that for each $a \in \underline{I}_X$,

$$(x_j, x) \Vdash_X a$$

for all $j \in \underline{J}$ with $\beta(a) \preccurlyeq_J j$. We then write $\boldsymbol{x} \to x$, and x is called a limit of $\boldsymbol{x}$.

Fundamental definitions

A net $(x_j)_{j \in \underline{J}} \in \underline{X}^J$ is a Cauchy net in X if there exists $\alpha \in \hom(I_X, J)$, called a (Cauchy) modulus, such that for each $a \in \underline{I}_X$,

$$(x_j, x_{j'}) \Vdash_X a$$

for all $j, j' \in \underline{J}$ with $\alpha(a) \preccurlyeq_J j, j'$.

For each $x \in \underline{X}$, the constant function $j \mapsto x$ from $\underline{J}$ into $\underline{X}$, denoted by $(x)_{j \in \underline{J}}$, is a Cauchy net in X with any modulus $\alpha \in \hom(I_X, J)$.

Lemma 2

Every convergent net in a uniform space is a Cauchy net.

Fundamental definitions

Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively. Then a function $f : \underline{X} \to \underline{Y}$ is uniformly continuous if there exists $\gamma \in \hom(I_Y, I_X)$, called a modulus (of uniform continuity), such that for each $b \in \underline{I}_Y$,

$$(x, y) \Vdash_X \gamma(b) \Rightarrow (f(x), f(y)) \Vdash_Y b$$

for all $x, y \in \underline{X}$.

A uniformly continuous mapping $f : X \to Y$ is

- a uniform isomorphism if there exists a uniformly continuous mapping $g : Y \to X$, called an inverse of f, such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$;
- X and Y are uniformly equivalent if there exists a uniform isomorphism between X and Y;
- we then write $X \simeq Y$.

Fundamental definitions

Lemma 3

Let X and Y be uniform spaces. Then every uniformly continuous function $f : \underline{X} \to \underline{Y}$ is a setoid mapping.

Lemma 4

Let X and Y be uniform spaces, and let J be a directed preordered set. Then for each uniformly continuous function $f : \underline{X} \to \underline{Y}$ and each net $\boldsymbol{x} \in \underline{X}^J$,

1. *for all $x \in \underline{X}$, if $\boldsymbol{x} \to x$ in X, then $f \circ \boldsymbol{x} \to f(x)$ in Y;*
2. *if $\boldsymbol{x}$ is a Cauchy net in X, then $f \circ \boldsymbol{x}$ is a Cauchy net in Y.*

Fundamental definitions

Proposition 5

Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively, and define $\rho_{X\times Y} \in \mathrm{hom}(I_X \times I_Y, I_X \times I_Y)$ and a relation $\Vdash_{X\times Y}$ between $(\underline{X} \times \underline{Y}) \times (\underline{X} \times \underline{Y})$ and $\underline{I}_X \times \underline{I}_Y$ by

$$\rho_{X\times Y} = \rho_X \times \rho_Y,$$
$$((x, y), (x', y')) \Vdash_{X\times Y} (a, b) \Leftrightarrow (x, x') \Vdash_X a \text{ and } (y, y') \Vdash_Y b$$

for all $(a, b) \in \underline{I}_X \times \underline{I}_Y$ and $(x, y), (x', y') \in \underline{X} \times \underline{Y}$. Then $(I_X \times I_Y, \rho_{X\times Y}, \Vdash_{X\times Y})$ is a uniform structure on the product setoid $X \times Y$.
A uniform space $X \times Y$ with the uniform structure is called the product of uniform spaces X and Y.

Completeness

Lemma 6

Let $X = (\underline{X}, =_X)$ *be a uniform space with a uniform structure* $(I_X, \rho_X, \Vdash_X)$*, and let* $J = (\underline{J}, \preccurlyeq_J)$ *be a directed preordered set. Then a binary relation* $=_{X^J}$ *on* $\underline{X}^J$ *given by*

$$\boldsymbol{x} =_{X^J} \boldsymbol{y} \Leftrightarrow \forall a \in \underline{I}_X \, \exists j \in \underline{J} \forall i \in \underline{J} \, (j \preccurlyeq_J i \Rightarrow (x_i, y_i) \Vdash_X a)$$

for all $\boldsymbol{x} = (x_i)_{i \in \underline{J}}, \boldsymbol{y} = (y_i)_{i \in \underline{J}} \in \underline{X}^J$*, is an equivalence relation on* $\underline{X}^J$*; hence* $X^J = (\underline{X}^J, =_{X^J})$ *is a setoid.*

Completeness

Proposition 7

Let $X = (\underline{X}, =_X)$ be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$, and let $J = (\underline{J}, \preccurlyeq_J)$ be a directed preordered set. Then a relation $\Vdash_{X^J}$ between $\underline{X}^J \times \underline{X}^J$ and $\underline{I}_X$ given by

$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^J} a \Leftrightarrow \exists \boldsymbol{x}', \boldsymbol{y}' \in \underline{X}^J [\boldsymbol{x} =_{X^J} \boldsymbol{x}' \wedge \boldsymbol{y} =_{X^J} \boldsymbol{y}' \\ \wedge \exists j \in \underline{J} \forall i \in \underline{J} (j \preccurlyeq_J i \Rightarrow (x'_i, y'_i) \Vdash_X a)]$$

for all $a \in \underline{I}_X$ and $\boldsymbol{x}, \boldsymbol{y} \in \underline{X}^J$ where $\boldsymbol{x}' = (x'_i)_{i \in \underline{J}}$ and $\boldsymbol{y}' = (y'_i)_{i \in \underline{J}}$, gives a uniform structure $(I_X, \rho_X^2, \Vdash_{X^J})$ on $X^J = (\underline{X}^J, =_{X^J})$; hence X^J is a uniform space.

Completeness

Lemma 8

Let X be a uniform space, and let J be a directed preordered set. Then a function $\eta_X^J : \underline{X} \to \underline{X}^J$, given by

$$\eta_X^J : x \mapsto (x)_{j \in \underline{J}}$$

for all $x \in \underline{X}$, is a uniformly continuous setoid injection such that

$$\eta_X^J \circ \boldsymbol{x} \to \boldsymbol{x}$$

in X^J for all Cauchy net $\boldsymbol{x} \in \underline{X}^J$.

Completeness

Let X be a uniform space with a uniform structure $(I_X, \rho_X, \Vdash_X)$. Then a regular net in X is a Cauchy net on the directed preordered set I_X with a modulus $\mathrm{id}_{\underline{I_X}} \in \mathrm{hom}(I_X, I_X)$.

Let $\tilde{X}$ be a setoid consisting of a set $\underline{\tilde{X}}$ and an equivalence relation $=_{\tilde{X}}$ given by

$$\underline{\tilde{X}} = \{\boldsymbol{x} \in \underline{X}^{I_X} \mid \boldsymbol{x} \text{ is a regular net}\}, \qquad \boldsymbol{x} =_{\tilde{X}} \boldsymbol{y} \Leftrightarrow \boldsymbol{x} =_{X^{I_X}} \boldsymbol{y}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \underline{\tilde{X}}$, respectively. Then $\tilde{X}$ with a uniform structure $(I_X, \rho_{X^{I_X}}, \Vdash_{\tilde{X}})$ is called a completion of X, where

$$(\boldsymbol{x}, \boldsymbol{y}) \Vdash_{\tilde{X}} a \Leftrightarrow (\boldsymbol{x}, \boldsymbol{y}) \Vdash_{X^{I_X}} a$$

for all $a \in \underline{I_X}$ and $\boldsymbol{x}, \boldsymbol{y} \in \underline{\tilde{X}}$; we then write η_X for $\eta_X^{I_X} : X \to \tilde{X}$.

Completeness

A uniform space X is complete if $\eta_X : X \to \tilde{X}$ is a uniform isomorphism.

Proposition 9

Every Cauchy net in a complete uniform space converges.

Theorem 10

The completion $\tilde{X}$ of a uniform space X is complete.

Theorem 11

Let X and Y be uniform spaces. Then

$$\tilde{X} \times \tilde{Y} \simeq \widetilde{X \times Y},$$

and $X \times Y$ is complete whenever so are X and Y.

Completeness

Let X and Y be uniform spaces with uniform structures $(I_X, \rho_X, \Vdash_X)$ and $(I_Y, \rho_Y, \Vdash_Y)$, respectively. Then a function $f : \underline{X} \to \underline{Y}$ is locally uniformly continuous if there exists a function $\boldsymbol{z} \mapsto \gamma_{\boldsymbol{z}}$ from $\underline{\tilde{X}}$ into $\hom(I_Y, I_X)$, called a family of local moduli, such that for each $b \in \underline{I}_Y$,

$$(\boldsymbol{z}, \eta_X(x)) \Vdash_{\tilde{X}} \gamma_{\boldsymbol{z}}(b) \text{ and } (\boldsymbol{z}, \eta_X(y)) \Vdash_{\tilde{X}} \gamma_{\boldsymbol{z}}(b) \Rightarrow (f(x), f(y)) \Vdash_Y b$$

for all $\boldsymbol{z} \in \underline{\tilde{X}}$ and $x, y \in \underline{X}$.

Completeness

Lemma 12

Let X and Y be uniform spaces. Then every locally uniformly continuous function $f : \underline{X} \to \underline{Y}$ is a setoid mapping.

Lemma 13

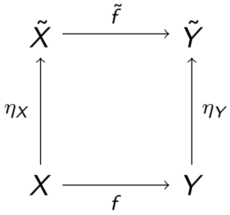
Let X and Y be uniform spaces, and let J be a directed preordered set. Then for each locally uniformly continuous function $f : \underline{X} \to \underline{Y}$ and each net $\boldsymbol{x} \in \underline{X}^J$,

1. *for all $x \in \underline{X}$, if $\boldsymbol{x} \to x$, then $f \circ \boldsymbol{x} \to f(x)$;*
2. *if $\boldsymbol{x}$ is a Cauchy net, then $f \circ \boldsymbol{x}$ is a Cauchy net.*

Completeness

Theorem 14

Let X and Y be uniform spaces. Then for each uniformly continuous (respectively, locally uniformly continuous) function $f : X \to Y$, there exists a uniformly continuous (respectively, locally uniformly continuous) function $\tilde{f} : \underline{\tilde{X}} \to \underline{\tilde{Y}}$ which makes the following diagram commute, that is, $\tilde{f} \circ \eta_X \sim \eta_Y \circ f$. Furthermore, such an $\tilde{f}$ is unique in the sense that for each locally uniformly continuous mapping $h : \tilde{X} \to \tilde{Y}$, if $h \circ \eta_X \sim \eta_Y \circ f$, then $h \sim \tilde{f}$.



Topological vector spaces and lattices

Topological vector spaces

A vector space (over $\mathbb{R}$) is a setoid $X = (\underline{X}, =_X)$ equipped with a setoid mapping $(x, y) \mapsto x + y$ of $X \times X$ into X, called addition, a setoid mapping $x \mapsto -x$ of X into X, called inverse, a setoid mapping $(s, x) \mapsto sx$ of $\mathbb{R} \times X$ into X, called scalar multiplication and an element 0 of $\underline{X}$, called the zero element, such that

$$
\begin{aligned}
(x + y) + z &=_X x + (y + z), & x + y &=_X y + x, \\
x + 0 &=_X x, & x + (-x) &=_X 0, \\
s(x + y) &=_X sx + sy, & (s + t)x &=_X sx + tx, \\
s(tx) &=_X (st)x, & 1x &=_X x
\end{aligned}
$$

for all $x, y, z \in \underline{X}$ and $s, t \in \underline{\mathbb{R}}$.

Example 15

Let $\underline{F}[0,1]$ be the set of setoid mappings of $[0,1]$ into $\mathbb{R}$. Then the setoid $F[0,1] = (\underline{F}[0,1], \sim)$ is a vector space equipped with addition, inverse, scalar multiplication and zero element given by

$$\begin{aligned} (f+g)(x) &= f(x)+g(x), & (-f)(x) &= -f(x), \\ (sf)(x) &= sf(x), & 0(x) &= 0 \end{aligned}$$

for all $f, g \in \underline{F}[0,1]$, $s \in \mathbb{R}$ and $x \in [0,1]$.

Topological vector spaces

Let X is a vector space. Then a linear functional on X is a setoid mapping $f : X \to \mathbb{R}$ such that

$$f(x+y) =_{\mathbb{R}} f(x) + f(y) \qquad \text{and} \qquad f(sx) =_{\mathbb{R}} sf(x)$$

for all $x, y \in \underline{X}$ and $s \in \underline{\mathbb{R}}$.

Topological vector spaces

A topological vector space is a vector space X equipped with a uniform structure $(I_X, \rho_X, \Vdash_X)$ such that

1. the addition $+ : X \times X \to X$ is uniformly continuous;
2. there exists a function $\xi^X : \underline{I}_X \times \underline{X} \to \mathbb{N}$ such that for each $a \in \underline{I}_X$,
$$(0, sx) \Vdash_X a$$
for all $x \in \underline{X}$ and $s \in \underline{\mathbb{R}}$ with $|s| \leq_{\mathbb{R}} 2^{-\xi(a,x)}$;
3. for each $a \in \underline{I}_X$,
$$(0, x) \Vdash_X a \Rightarrow (0, sx) \Vdash_X a$$
for all $x \in \underline{X}$ and $s \in \underline{\mathbb{R}}$ with $|s| \leq_{\mathbb{R}} 1$.

Topological vector spaces

Lemma 16

Let X be a topological vector space. Then the inverse $x \mapsto -x$ is uniformly continuous, and the scalar multiplication $(s, x) \mapsto sx$ is locally uniformly continuous.

Theorem 17

If X is a topological vector space, then so is its completion $\tilde{X}$.

Topological vector lattices

A (join) semilattice is a setoid $X = (\underline{X}, =_X)$ equipped with a setoid mapping $(x, y) \mapsto x \vee y$ of $X \times X$ into X, called a join, such that

$$x \vee (y \vee z) =_X (x \vee y) \vee z, \quad x \vee y =_X y \vee x, \quad x \vee x =_X x$$

for all $x, y, z \in \underline{X}$.

Let $X = (\underline{X}, =_X)$ be a semilattice. Then the (canonical) partial order $\leq_X$ on X is given by

$$x \leq_X y \Leftrightarrow x \vee y =_X y$$

for all $x, y \in \underline{X}$, and $x \vee y$ is the least upper bound of $\{x, y\}$ for all $x, y \in \underline{X}$.

Topological vector lattices

A vector lattice is a vector space $X = (\underline{X}, =_X)$ such that

1. X is a semilattice with a join $\vee$;
2. $(x + z) \vee (y + z) =_X x \vee y + z$;
3. if $0 \leq_{\mathbb{R}} s$, then $s(x \vee y) =_X (sx) \vee (sy)$

for all $x, y, z \in \underline{X}$ and $s \in \underline{\mathbb{R}}$.

Lemma 18
Let $X = (\underline{X}, =_X)$ be a vector lattice. Then

1. *if $x \leq_X y$, then $x + z \leq_X y + z$;*
2. *if $x \leq_X y$ and $0 \leq_{\mathbb{R}} s$, then $sx \leq_X sy$*

for all $x, y, z \in \underline{X}$ and $s \in \underline{\mathbb{R}}$.

Topological vector lattices

Example 19

Note that $F[0,1]$ in Example 15 is a vector lattice with a join given by

$$(f \vee g)(x) = \max_{\mathbb{R}}(f(x), g(x))$$

for all $f, g \in \underline{F}[0,1]$ and $x \in [0,1]$. Let $\underline{C}[0,1]$ be a set given by

$$\underline{C}[0,1] = \{(f,\gamma) \in \underline{F}[0,1] \times \hom(\mathbb{N},\mathbb{N}) \mid f \text{ is uniformly continuous with a modulus } \gamma\},$$

and let $=_{C[0,1]}$ be an equivalence relation on $\underline{C}[0,1]$ given by

$$(f, \gamma^f) =_{C[0,1]} (g, \gamma^g) \Leftrightarrow f =_{F[0,1]} g.$$

Then the setoid $C[0,1] = (\underline{C}[0,1], =_{C[0,1]})$ is a vector lattice.

Topological vector lattices

Let $X = (\underline{X}, =_X)$ be a vector lattice with a join $\vee : X \times X \to X$. Then a meet $\wedge : X \times X \to X$ is given by

$$x \wedge y = -(-x \vee -y)$$

for all $x, y \in \underline{X}$.

Proposition 20

Every vector lattice $X = (\underline{X}, =_X)$ is a distributive lattice, that is,
$x \vee (y \wedge z) =_X (x \vee y) \wedge (x \vee z)$*, or*
$x \wedge (y \vee z) =_X (x \wedge y) \vee (x \wedge z)$ *for all* $x, y, z \in \underline{X}$*.*

Topological vector lattices

Let X be a vector lattice. Then the subset

$$\underline{C}_X = \{x \in \underline{X} \mid 0 \leq_X x\}$$

is called a positive cone of X; note that $C_X = (\underline{C}_X, \leq_X)$ with an upper bound function

$$\vee \in \mathrm{hom}(\underline{C}_X \times \underline{C}_X, \underline{C}_X)$$

is a directed preordered set.

A linear functional f on X is positive if $0 \leq_{\mathbb{R}} f(x)$ for all $x \in \underline{C}_X$.

Example 21

Let $R : \underline{C}[0,1] \to \mathbb{R}$ be a function given by

$$R(f,\gamma) = \int f,$$

where $\int$ is the Riemann integral, for all $(f,\gamma) \in \underline{C}[0,1]$. Then R is a positive linear functional.

Topological vector lattices

Let $X = (\underline{X}, =_X)$ be a vector lattice, and let $(\text{-})^+ : X \to X$, $(\text{-})^- : X \to X$ and $|\text{-}| : X \to X$ be setoid mappings given by

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x \vee (-x),$$

respectively, for all $x \in \underline{X}$; note that $x^+, x^- \in \underline{C}_X$.

A vector lattice X is Archimedean if for each $x \in \underline{X}$, $x \leq_X 0$ whenever there exists $y \in \underline{X}$ such that $x \leq_X 2^{-n} y$ for all $n \in \mathbb{N}$.

Lemma 22

Let $X = (\underline{X}, =_X)$ be an Archimedean vector lattice. Then

1. *if* $0 \leq_X x$, *then* $\max_{\mathbb{R}}(s, t)x =_X sx \vee tx$;
2. $|sx| =_X |s||x|$

for all $s, t \in \mathbb{R}$ *and* $x \in \underline{X}$.

Topological vector lattices

A topological vector lattice is a vector lattice X equipped with a uniform structure $(I_X, \rho_X, \Vdash_X)$ such that

1. X is a topological vector space with the uniform structure;
2. the join $\vee : X \times X \to X$ is uniformly continuous;
3. for each $a \in \underline{I}_X$,

$$(0, y) \Vdash_X a \Rightarrow (0, x) \Vdash_X a$$

   for all $x, y \in \underline{C}_X$ with $x \leq_X y$.

Topological vector lattices

Lemma 23

Every topological vector lattice is Archimedean.

Theorem 24

If X is a topological vector lattice, then so is its completion $\tilde{X}$.

Integration theory

Integration theory

An abstract integration space is a vector lattice X equipped with a positive linear functional E on X.

Example 25

$(C[0,1], R)$ is an abstract integration space.

In the following, we fix an abstract integration space (X, E).

Integrable functions

Lemma 26

Let $=_L$ *be a binary relation on* $\underline{X}$ *given by*

$$x =_L y \Leftrightarrow \forall n \in \mathbb{N}\,(E(|x-y|) \leq_{\mathbb{R}} 2^{-n})$$

for all $x, y \in \underline{X}$*. Then* $=_L$ *is an equivalence relation on* $\underline{X}$*; hence* $L = (\underline{X}, =_L)$ *is a setoid.*

Proposition 27

Let ρ_L *and* $\Vdash_L$ *be a monotone function from* $\mathbb{N}$ *into* $\mathbb{N}$ *and a relation between* $\underline{X} \times \underline{X}$ *and* $\mathbb{N}$ *defined by*

$$\rho_L(n) = n+1 \qquad \text{and} \qquad (x, y) \Vdash_L n \Leftrightarrow E(|x-y|) \leq_{\mathbb{R}} 2^{-n},$$

respectively, for all $n \in \mathbb{N}$ *and* $(x, y) \in \underline{X} \times \underline{X}$*. Then* $(\mathbb{N}, \rho_L, \Vdash_L)$ *is a uniform structure on the setoid* $L = (\underline{X}, =_L)$*.*

Integrable functions

Proposition 28

L is a topological vector lattice.

We write $\mathfrak{L}$ for the completion $\tilde{L}$ of the topological vector lattice L, and call an element of $\mathfrak{L}$ an integrable function over the abstract integration space (X, E).

Integrable functions

Proposition 29

There exists a uniformly continuous mapping $\int : \mathfrak{L} \to \mathbb{R}$ *such that*

1. $\int \eta_L(x) =_{\mathbb{R}} E(x)$,
2. $\int (f + g) =_{\mathbb{R}} \int f + \int g$,
3. $\int (sf) =_{\mathbb{R}} s \int f$,
4. *if* $0 \leq_{\mathfrak{L}} f$, *then* $0 \leq_{\mathbb{R}} \int f$

for all $x \in \underline{L}$, $f, g \in \underline{\mathfrak{L}}$ *and* $s \in \underline{\mathbb{R}}$. *For* $f \in \underline{\mathfrak{L}}$, $\int f$ *is called the* *integral* *of* f.

Measurable functions

Lemma 30

Let $=_M$ be a binary relation on $\underline{X}$ given by

$$x =_M y \Leftrightarrow \forall u \in \underline{C}_X \, \forall n \in \mathbb{N} \, (E(|x - y| \wedge u) \leq_{\mathbb{R}} 2^{-n})$$

for all $x, y \in \underline{X}$. Then $=_M$ is an equivalence relation on $\underline{X}$; hence $M = (\underline{X}, =_M)$ is a setoid.

Proposition 31

Let ρ_M and $\Vdash_M$ be a monotone function from $C_X \times \mathbb{N}$ into $C_X \times \mathbb{N}$, and a relation between $\underline{X} \times \underline{X}$ and $\underline{C}_X \times \mathbb{N}$, respectively, defined by

$$\rho_M(u, n) = (u, n + 1)$$
$$(x, y) \Vdash_M (u, n) \Leftrightarrow E(|x - y| \wedge u) \leq_{\mathbb{R}} 2^{-n},$$

respectively, for all $(u, n) \in \underline{C}_X \times \mathbb{N}$ and $(x, y) \in \underline{X} \times \underline{X}$. Then $(C_X \times \mathbb{N}, \rho_M, \Vdash_M)$ is a uniform structure on the setoid $M = (\underline{X}, =_M)$.

Measurable functions

Proposition 32

M is a topological vector lattice.

We write $\mathfrak{M}$ for the completion $\tilde{M}$ of the topological vector lattice M, and call an element of $\mathfrak{M}$ a measurable function over the abstract integration space (X, E).

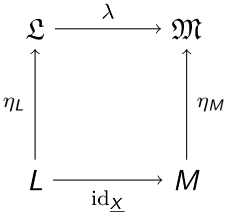
Convergence theorems

Lemma 33

The function $\mathrm{id}_{\underline{X}} : \underline{X} \to \underline{X}$ *is a uniformly continuous setoid injection of* L *into* M*.*

Proposition 34

There exists a uniformly continuous embedding $\lambda : \mathfrak{L} \to \mathfrak{M}$ *such that* $\eta_M \circ \mathrm{id}_{\underline{X}} \sim \lambda \circ \eta_L$*.*



Convergence theorems

Lemma 35

The function $(\text{-}) \wedge |\text{-}| : (x, y) \mapsto x \wedge |y|$ *from* $\underline{X} \times \underline{X}$ *into* $\underline{X}$ *is a locally uniformly continuous mapping of* $L \times M$ *into* L*.*

Proposition 36

There exists a locally uniformly continuous mapping $\mu : \mathfrak{L} \times \mathfrak{M} \to \mathfrak{L}$ *such that* $\mu \circ (\eta_L \times \eta_M) \sim \eta_L \circ ((\text{-}) \wedge |\text{-}|)$.

$$\begin{array}{ccc} \mathfrak{L} \times \mathfrak{M} & \xrightarrow{\mu} & \mathfrak{L} \\ {\scriptstyle \eta_L \times \eta_M} \uparrow & & \uparrow {\scriptstyle \eta_L} \\ L \times M & \xrightarrow[(\text{-}) \wedge |\text{-}|]{} & L \end{array}$$

Convergence theorems

Lemma 37

For all $f, g \in \mathfrak{L}$ and $h \in \mathfrak{M}$,

$$\mu(g, \lambda(f)) =_{\mathfrak{L}} g \wedge |f| \qquad \text{and} \qquad \lambda(\mu(g, h)) =_{\mathfrak{M}} \lambda(g) \wedge |h|.$$

Theorem 38

Let f be a measurable function. If there exists an integrable function g such that $|f| \leq_{\mathfrak{M}} \lambda(g)$, then there exists an integrable function $f_{\mathfrak{L}}$ such that $f =_{\mathfrak{M}} \lambda(f_{\mathfrak{L}})$.

Convergence theorems

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions, and let f be an integrable function. Then

1. $(f_n)_{n\in\mathbb{N}}$ converges in norm to f if $f_n \to f$ in $\mathfrak{L}$;
2. $(f_n)_{n\in\mathbb{N}}$ converges in measure to f if $\lambda(f_n) \to \lambda(f)$ in $\mathfrak{M}$.

Theorem 39 (Lebesgue's Monotone Convergence Theorem)

Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of integrable functions. Then the following are equivalent.

1. *$(f_n)_{n\in\mathbb{N}}$ converges in measure to some integrable function f,*
2. *$(f_n)_{n\in\mathbb{N}}$ converges in norm to some integrable function f,*
3. *$(\int f_n)_{n\in\mathbb{N}}$ converges;*

in which case $\int f_n \to \int f$.

Convergence theorems

Theorem 40 (Fatou's Lemma)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions converging in measure to an integrable function f such that $0 \leq_{\mathfrak{L}} f_n$ and $\int f_n \leq A$ for all $n \in \mathbb{N}$. Then $\int f \leq A$.

Theorem 41 (Lebesgue's Dominated Convergence Theorem)

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions converging in measure to an integrable function f, and let g be an integrable function such that $|f_n| \leq_{\mathfrak{L}} g$ for all $n \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$ converges in norm to f.

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