On the compatibility of dependent type theory with Brouwer's Intuitionistic principles

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(based on joint work with P. Sabelli)

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A workshop in honor of Martin Escardò 's 60th birthday

Abstract

- Bishop's Compatibility issues about constructive maths

- Compatibility/Incompatibility with Brouwer's Intuitionistic maths

  of type theoretic foundations for constructive maths

- Open problem about Brouwer's intuitionistic principles

Brouwer's intuitionistic Mathematics

intuitionistic mathematics by **Brouwer** 

is **Incompatible with classical mathematics**

because of **Brouwer**'s **continuity principles**

implying that

all functions between real numbers must be continuous!

Brouwer's intuitionistic Mathematics

Brouwer's **intuitionistic continuity principles**: 

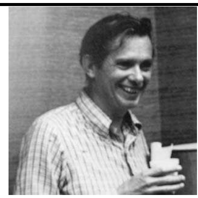
Bar Induction (**BI**) =**spatiality** of **Baire locale**

(⇒ we can reason **inductively** **in the Baire space**)

Local Continuity Principle (**LCP**)= **continuity of all functions** from **Baire space** to **Nat**

where **Brouwer**'s **choice sequences**= **functional relations**

Constructive Mathematics between intuitionistic and classical one

according to Errett Bishop 

author of "Foundations of Constructive Analysis" 1967

constructive maths must be compatible
both with classical and intuitionistic mathematics

constructive maths

classical

intuitionistic

Constructive Mathematics by Bishop

Bishop 

showed in "Foundations of constructive analysis"

that a large portion of functional analysis

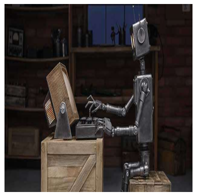
can be reproduced constructively

with constructive proofs

having a computational contents

so that the existence of an object

can be computed by a machine



and axiom of choice is valid

Essence of constructive mathematics

constructive mathematics

=

IMPLICIT computational mathematics

made EXPLICIT

in computable models

validating Church thesis CT

$+$

Axiom of choice AC

for program-extraction from proofs

at least on Natural Numbers

A principle of recursive mathematics: Church Thesis

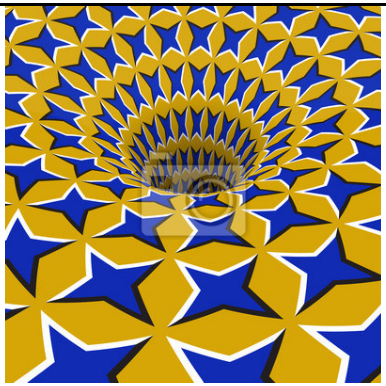
$$(\mathbf{CT}) \quad \frac{\forall x \in \mathsf{N}at\ \exists!\, y \in \mathsf{N}at\ \ R(x,y)}{\exists \mathbf{e} \in \mathsf{N}at \quad (\,\forall x \in \mathsf{N}at\ \exists y \in \mathsf{N}at\ \ T(\mathbf{e},x,y)\ \ \&\ \ R(x,U(y))\,)\,)}$$

functional relations are all **computable**

valid in recursive constructive maths

of Russian constructivism initiated by **Markov**

Number-theoretic **Axiom of choice**



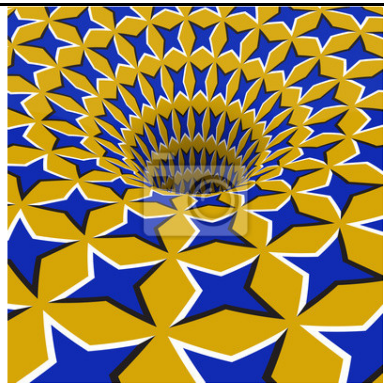
$(AC_{Nat,Nat})$

$$\forall x \in Nat\ \exists y \in Nat\ R(x,y) \quad \longrightarrow$$

$$\exists f \in Nat \to Nat\ \ \forall x \in Nat\ \ R(\,x\,,\ f(x)\,)$$

from any total relation we can extract a function

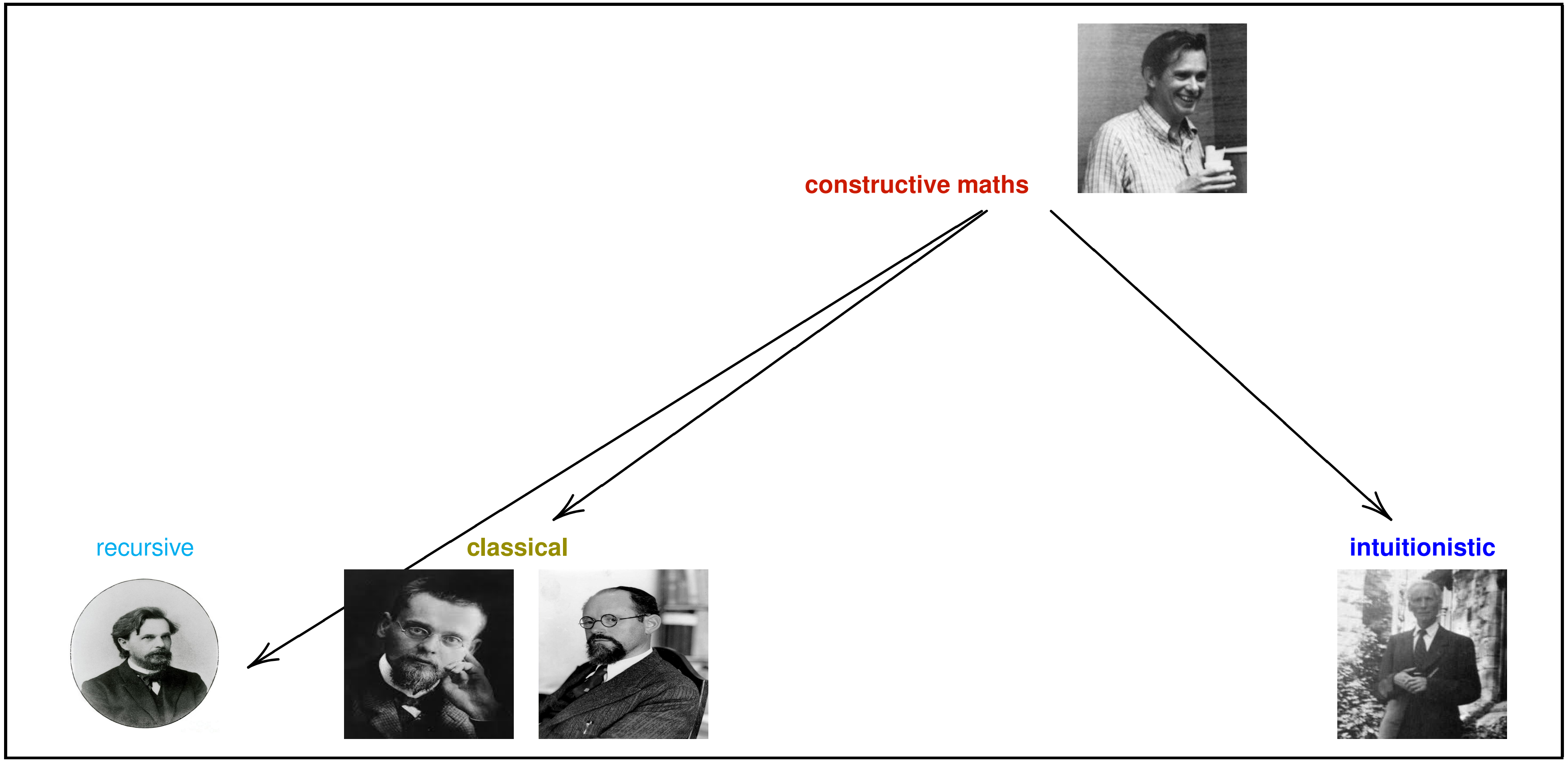
Axiom of choice



$$(AC) \quad \forall x \in A \; \exists y \in B \; R(x,y) \quad \longrightarrow \quad \exists f \in A \to B \;\; \forall x \in A \;\; R(\, x \,,\, f(x)\,)$$

from any total relation we can extract a function

Refinement: Constructive Maths among recursive, intuitionistic and classical ones



Brouwer's intuitionism incompatible with Russian constructivism

Brouwer's continuity principles + Church thesis $\vdash \perp$

($\Rightarrow$ incompatible with constructive recursive maths

of Russian constructivism including CT)

i.e. with because

$\mathbf{HA}^{\omega}$ $+$ Fan theorem $+$CT is inconsistent

where

Fan theorem = *spatiality* of Cantor space

consequence of BI

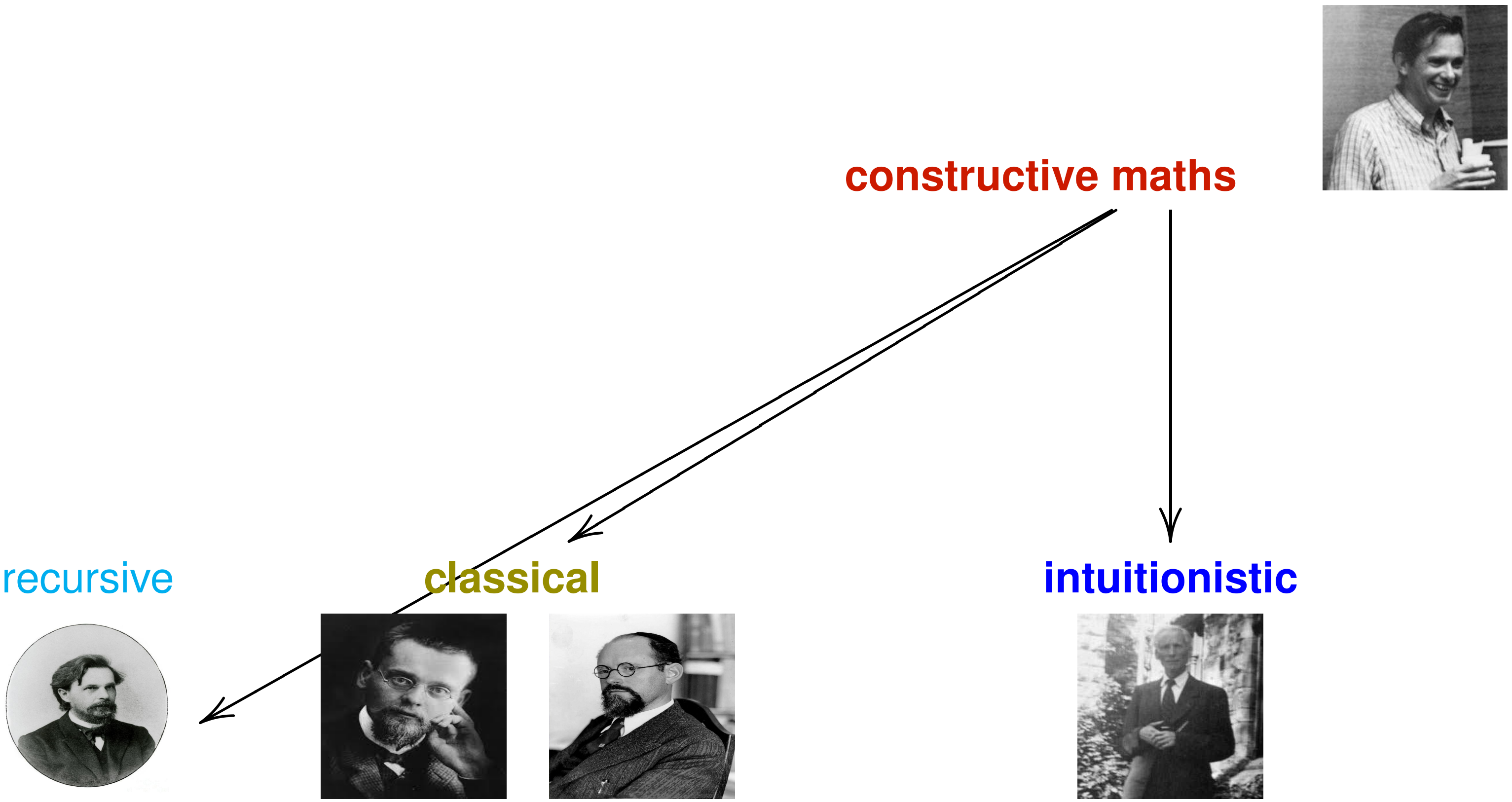
choice sequences=functional relations

What choice principles in Constructive Maths ?

What **choice principles**

can we use in **constructive maths**

to keep **Bishop**'s compatibility requirements?



Brouwer's LCP contradicts Axiom of Choice (AC) and extensionality

from Troelstra 1977

Heyting arithmetics with finite types $+$ LCP $+$ AC+ extfun $\vdash \bot$

where

$$\text{extfun}\ \frac{f(x) =_B g(x)\ \ true\ \ [x \in A]}{\lambda x.f(x) =_{A \to B} \lambda x.g(x)\ \ true}$$

extensionality of functions

Brouwer's LCP contradicts Axiom of Choice (AC) contradicts the ξ-rule

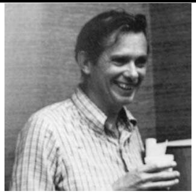
from Escardò-Xu 2015

Martin-Löf's type theory $+$ LCP $+$ AC$+$ ξ-rule $\vdash \bot$

where

$$\xi\text{-rule}\ \frac{f(x) = g(x) \in B(x) \quad [x \in A]}{\lambda x.f(x) = \lambda x.g(x) \in \Pi_{x \in A} B(x)}$$

What foundation for constructive mathematics ??



Since the 80s various foundations for **Bishop's constructive mathematics appeared including**

- **Martin-Löf's type theory** - incompatible with **Brouwer**'s **intuitionistic maths**
- **Aczel set theory** - incompatible with CT +**axiom of choice**
- **Homotopy type theory** - incompatible with CT +**axiom of choice**

. . .

Church Thesis CT + Axiom of Choice AC contradicts extensionality

Heyting arithmetics with finite types $+ \, CT + \mathbf{AC} + $ extfun $\vdash \bot$

Why a minimalist foundation for constructive mathematics ??

j.w.w. Giovanni Sambin

We wanted to take advantage of the plurality

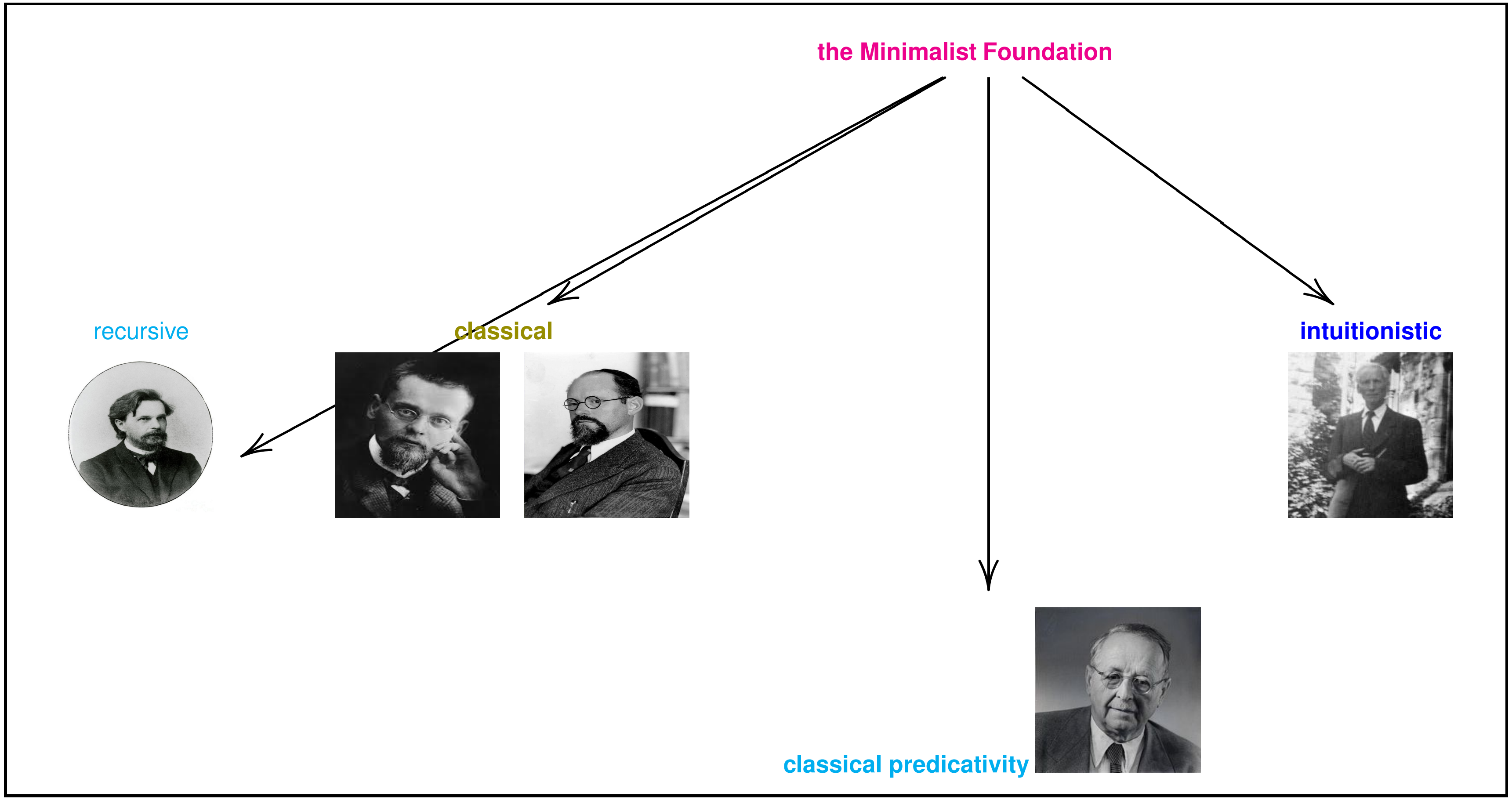
of **foundations** for **Bishop's constructive mathematics**

Plurality of foundations ⇒ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand-Huet's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory **HoTT** and Voevodsky's Univalent Foundations Feferman's constructive expl. maths

the **Minimalist Foundation MF**

Peculiarity of the Minimalist Foundation for constructive maths



Our TWO-LEVEL Minimalist Foundation

from [Maietti'09] in agreement with [M. Sambin2005]

its intensional level mTT

(Minimalist Type Theory)

= a PREDICATIVE VERSION of Coquand-Huet's Calculus of Constructions CC

(fragment of Rocq)

(consistent with CT + AC)

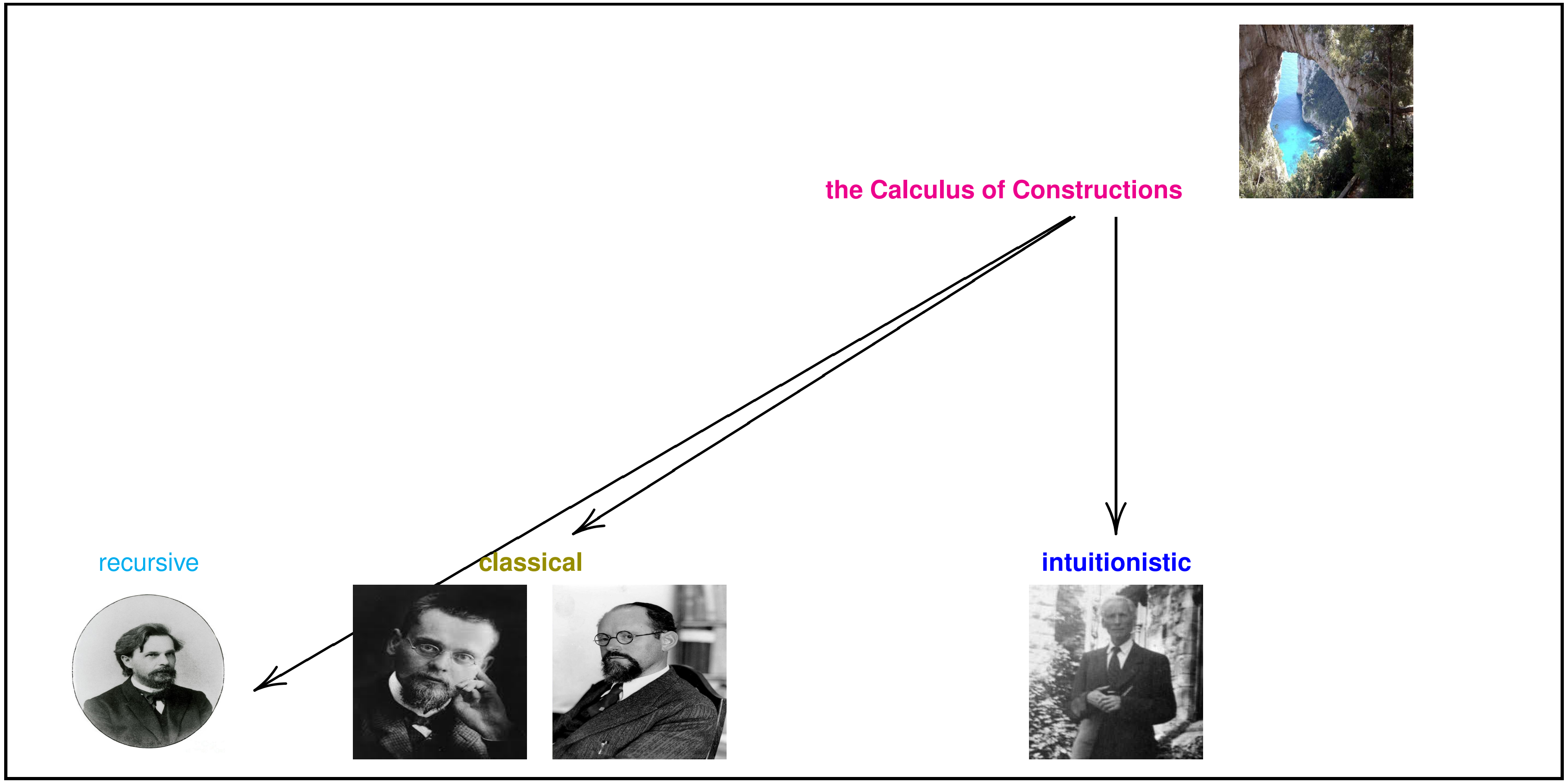
its extensional level emTT

(extensional Minimalist Type Theory)

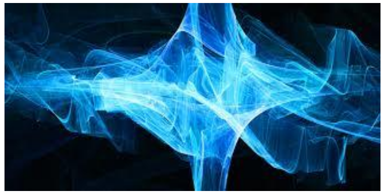
is a PREDICATIVE LOCAL set theory

(NO choice principles)

Peculiarity of the Calculus of Constructions for Rocq



two notions of function in the Minimalist Foundation



a *primitive notion* of **type-theoretic function**

$$f(x) \in B \; [x \in A]$$

(closed under ”exponentiation”)

$\neq$ **(syntactically)**

notion of functional relation

$$\forall x \in A \; \exists! y \in B \; R(x, y)$$

(NOT closed under ”exponentiation”)

Axiom of unique choice

$$\forall x \in A\, \exists! y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

Type-theoretic Church thesis

(**TCT**) $\forall f \in \mathsf{N}at \rightarrow \mathsf{N}at \quad \exists e \in \mathsf{N}at$

$$(\forall x \in \mathsf{N}at \;\; \exists y \in \mathsf{N}at \;\; T(e, x, y) \; \& \; U(y) =_{\mathsf{N}at} f(x))$$

type-theoretic functions ($\subset_{\neq}$ **functional relations**)

are all **computable**

Peculiariy of MF: reconciling Russian constructivism with Brouwer intuitionism

Theorem (joint with P. Sabelli):

Both levels of MF are consistent with $+$Theoretical Church Thesis$(\mathbf{TCT})$

+ Brouwer's continuity principles

+ Zermelo's AC$^{Nat}_{Baire}$ (extracting a functional relation)

Bar Induction (BI) =spatiality of Baire locale

Local Continuity Principle (LCP)= continuity of functions from Baire space to Nat

where Brouwer's choice sequences= functional relations

Proof: a model made of Hyland's assemblies formalized in

Aczel's CZF+REA + BI +LCP

(= a predicative and constructive quasi-topos of assemblies!!!)

Decomposition of Church Thesis

CT
$\Updownarrow$
TCT $+$ AC!$_{Nat,Nat}$

Axiom of unique choice

$$\forall x \in A\ \exists! y \in B\ R(x,y) \longrightarrow \exists f \in A \rightarrow B\ \forall x \in A\ R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

valid in Homotopy Type Theory, Martin-Löf's type theory and Aczel's CZF

but NOT in the Minimalist Foundation

Peculiariy of CC: reconciling Russian constructivism with Brouwer intuitionism

Theorem (joint with P. Sabelli):

CC is consistent with $+$Theoretical Church Thesis$(\mathbf{TCT})$

+ Brouwer's continuity principles BI +LCP

+ Zermelo's AC$_{Baire}^{Nat}$ (extracting a functional relation)

where **Brouwer's choice sequences= functional relations** 

Proof. **a model is given by Hyland's assemblies formalized in**

IZF + BI +LCP

(= an intuitionistic quasi-topos of assemblies!!!) in [M. & Sabelli & Trotta '25]

thanks to the equiconsistency of CC with its extensional version in [M. & Sabelli '24]

which is the internal language of arithmetic solid quasi-toposes

Open issues

from Troelstra 1977 we know that

HA^{ω} + AC +BI + LCP is consistent

+

Escardò-Xu's incompatibility of the ξ-rule with intuitionistic maths

⇓

-Is the intensional level of the Minimalist Foundation (which does not have the ξ-rule)

consistent with AC +BI + LCP ??

-Is Coquand-Huet's Calculus of Constructions / Martin-Löf's type theory

(both without ξ-rule) consistent with AC +BI + LCP ??