Quantifiers in Real Analysis

Based on jww Sam Sanders

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To Martín and his 60 years

with thanks for all the good and productive discussions we had.

Cantorian Set Theory

- Cantorian set theory offered two “tools” for mathematical constructions.
- The introduction of ordinals made transfinite recursion mathematically sound.
- Definitions involving higher order quantifiers and higher order parameters made it possible to study new classes of objects and describe operations with them as inputs.
- Example: Two proofs of the Cantor-Bendixson theorem for closed sets of reals.

The Kleene quantifiers

- $$\exists^2(f^1) = \begin{cases} 0 & \text{if} \quad \forall n(f(n) = 0) \\ 1 & \text{if} \quad \exists n(f(n) > 0) \end{cases}$$
- $$\exists^3(F^2) = \begin{cases} 0 & \text{if} \quad \forall f(F(f) = 0) \\ 1 & \text{if} \quad \exists f(F(f) > 0) \end{cases}$$
- These are clearly non-computable.
- The scopes are infinite.

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- For every closed ordinal $\alpha < \epsilon_0$ there is a closed subset X_α of $2^{\mathbb{N}}$, of order type α, such that the following quantifier $\exists_\alpha$, defined for arbitrary subsets Y of $2^{\mathbb{N}}$, is definable in Gödel's T:

$$\exists_\alpha(Y) = \begin{cases} 0 & \text{if} \quad X_\alpha \cap Y = \emptyset \\ 1 & \text{if} \quad X_\alpha \cap Y \neq \emptyset \end{cases}$$

- We say that X_α is *searchable*.

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- Then there is also a *selector* ν_α, definable in T, such that $\nu_\alpha(Y)$ is defined for all Y, and $\nu(Y) \in X_\alpha \cap Y$ when $X_\alpha \cap Y \neq \emptyset$.
- A set $X \subseteq 2^{\mathbb{N}}$ has to be closed and countable to be searchable or admit a computable selector, even with respect to PCF-definability or Kleene-computability.

ME + DN

- Martín conjectured, and DN proved, that when X is searchable in T, then the Cantor-Bendixson rank of X is bounded below ϵ_0.
- DN also proved that for any closed computable ordinal $\alpha < \omega_1^{\mathrm{CK}}$ there is a set $X_\alpha \subseteq 2^{\mathbb{N}}$ of order type α such that $\exists_\alpha$ is Kleene-computable.

A “new" theorem

Tying up some loose ends from the literature we even have

Theorem

Let $X \subseteq 2^{\mathbb{N}}$. The following are equivalent

1. *X is closed and countable*
2. *The functional $\exists_X$ is computable in some $f \in \mathbb{N}^{\mathbb{N}}$.*

Proof.

Exercise for those who like to tidy up mess. □

NOTE: $\exists_X$ is a functional of type 3.

What we learn from this

- The conclusion is that it is **possible** to have **effective** quantification over some infinite sets, but if we want to understand the complexity of quantification over sets that are not both countable and closed, we need tools from generalised computability theory (or something even more fancy).
- The aim of this talk is to describe some of these tools without going into technical details, and to explain why we are interested in them.

The NorSan-project

In my project with Sam (the NorSan project) we ask two foundational questions:

1. Given a theorem A of ordinary real analysis, what is the minimal set of axioms in Kohlenbach's higher-order theory for RM needed to prove A?
2. Given a construction in ordinary real analysis, what is the minimal set of non-computable tools we need to perform this construction?

For these questions to be precise, we need a basic theory or a basic notion of computability, but I will deliberately be vague on this. We will focus on 2.

Platek's thesis, a digression

- In his thesis, Platek proved the equivalence between Kleene's approach to HOC and an approach based on typed lambda-calculus (or combinators), see also Moldestad 1977.
- This (partly) inspired Scott to construct **LCF**, later transformed to **PCF** by Plotkin.
- As a technical tool, Platek considered an intermediate type structure consisting of partial functionals only defined on hereditarily total arguments.
- This has turned out to be the most relevant type structure for studying constructions in real analysis.

Non-computable computability

- We will be interested in what mathematicians will consider as constructions.
- These may involve discontinuous objects, or will be discontinuous by themselves.
- We include $\exists^2$ as a basic non-computable tool .
- We will then vary on which other functionals we add, and what model of computability we use, in our analysis of the complexity of certain constructions.
- For positive statements we can normally do with a small fragment of Gödel's T, while for negative statements we mostly prove them for a modernised version of the Kleene/Platek calculus.

On the foundation

- Our conception of the continuum is, as I see it, vague, as demonstrated by the results of Cohen and later use of forcing.
- Thus I see it as of foundational interest to understand how much comprehension involving quantifiers over the continuum we need for various constructions in real analysis, constructions involving third order parameters.

Gentle vs. violent representations

- Representations will, to some extent, always be needed.
- Known generalised computability models will not accept a discontinuous function $\phi : \mathbb{R} \to \mathbb{R}$ directly as an input.
- We gently lift ϕ to a $\hat{\phi} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, without adding any essential information. We do not think of these as *problematic representations*.

Gentle vs. violent representations

- This is in contrast to representations in the form of some $f \in \mathbb{N}^{\mathbb{N}}$.
- The implicit information in a representation of this kind will often be richer than what is available from the object represented directly, e.g. a modulus of continuity for a continuous function or a sequence of open sets intersecting to a $\mathbf{G}_\delta$-set.

Gentle vs. violent representations

- There is often an implicit theorem stating that a certain kind of object can be represented in a certain kind of way.
- Part of our project is to analyse such theorems in the style of reverse mathematics and the complexity of finding such representations in the tradition of HOC.

An illuminating example

Let Γ be the class of countable subsets of $\mathbb{R}$ (including the finite ones).

- If $X \in \Gamma$, then a representation of X will be an enumeration $\{x_i : i \in \mathbb{N}\}$ of X, or something close to this.
- From a selection function for Γ we can extract the well known example of a set that is not Lebesgue measurable.
- Consequently, a map selecting one representation of each element in Γ will need a substantial use of AC.

Type 0, 1 and 2

- This audience will recognise objects of type 0 or 1 when seen (order 1 or 2).
- Objects of type 2 will (in this talk) be subsets of the reals /$\mathbb{N}^{\mathbb{N}}$) when identified with their characteristic functions (predicates over the reals), functions from the reals to the reals etc.
- We will only consider (essentially) **total** objects of types 0, 1 or 2.

The functionals

Let Γ be a class of subsets X of $\mathbb{R}$, identified with their characteristic functions.

- Ω_Γ is the *partial* functional defined on Γ by

$$\Omega_\Gamma(X) = \begin{cases} 0 & \text{if} \quad X = \emptyset \\ 1 & \text{if} \quad X \neq \emptyset \end{cases}$$

- A *selector* for Γ is any ν_Γ, defined on Γ, such that $\nu_\Gamma(X) \in X$ whenever $X \in \Gamma$ is nonempty.

This corresponds to Escardó's search and selectors when Γ is the powerset of some subset Y of $\{0,1\}^{\mathbb{N}}$ (which we, when it suits us, consider as a subset of $\mathbb{R}$ via the Cantor set).

Cases studied (so far)

1. Γ_{b} of sets with at most one element.
2. Γ_{fin} of finite sets.
3. Γ_{C} of compact sets.
4. Γ_{count} of countable sets.
5. $\Gamma_{\mathbf{F}_\sigma}$ of $\mathbf{F}_\sigma$-sets.
6. $\Gamma_{\mathbf{G}_\delta}$ of $\mathbf{G}_\delta$-sets.
7. Γ_Δ of sets that are both $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$.
8. Γ_{scat} of scattered sets.

We write Ω_x for Ω_{Γ_x} when x is one of these eight cases. None of these are computable in any functional of type 2 and none are equivalent to a total functional.

The incorrectly named Ω_1

Let Γ_1 be the class of singletons $\{x\}$, identified with their characteristic functions.

Obviously Ω_{Γ_1} is trivial, being a subfunctional of the constant 1.

However, the one possible functional ν_{Γ_1} is non-trivial.

This turns **implicit** definitions into **explicit** ones.

ν_{Γ_1} was originally called Ω_1, and we stick to that name.

The fabulous Ω_{b}

- Now consider Ω_{b}, and let ν_{b} be the associated selector (with $\nu_{\mathrm{b}}(\emptyset) = 0$).
- Modulo $\exists^2$ we have that Ω_{b} and ν_{b} are computationally equivalent.
- Using the full power of Kleene computability one can show that Ω_{b} and Ω_{fin} are equivalent, but it is open if Ω_{fin} is definable from Ω_{b} and $\exists^2$ (or any other functional of type 2) via a term in system T.
- Ω_{b} is strictly stronger that Ω_1.

The power of Ω_b

- Ω_b was designed to compute the inverse of F on A from F and A, where $A \subset \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{N}$ is injective on A.
- This operator is equivalent to Ω_b.
- Ω_b is an oracle that enables us to define a set X, prove that X is finite, and then use that $X = \{x_1, \ldots, x_n\}$ when we continue a construction.

The power of Ω_b

- Functionals extracted from constructions in ordinary real analysis will often be computable in Ω_b if they involve a step from an implicit definition of a finite set to an explicit description of one.
- One key example is the Jordan decomposition of a function of bounded variation when the actual variation is not given.
- When Ω_b is computable in such functionals, this non-computable, but mathematically harmless, step is needed.

Theorems about Ω_{b}

Theorem

Ω_{b} is lame, i.e. if $f \in \mathbb{N}^{\mathbb{N}}$ is computable in g, $\exists^2$ and Ω_{b}, then f is hyperarithmetical in g.

Theorem

If Γ is a class of sets with $\emptyset \in \Gamma$, and Ω_Γ is computable in Ω_{b} and $\exists^2$, then there is a selector ν_Γ that is also computable in $\exists^2$ and Ω_{b}.

Corollary

Ω_{b} cannot “decide" if a countable set is empty or not.

The significance of Ω_b

- So why are the results on Ω_b of interest?
- Let us consider the Jordan decomposition theorem for real valued functions on $\mathbb{R}$ as an example.
- Our results show that the decomposition construction is genuinely of type 3, e.g. in contrast to integration of continuous functions.
- Analysing this construction in traditional computational analysis requires a way of representing the objects, a representation process that is itself of type 3.

The significance of Ω_b

- So why are the results on Ω_b of interest?
- Let us consider the Jordan decomposition theorem for real valued functions on $\mathbb{R}$ as an example.
- Our results show that the decomposition construction is genuinely of type 3, e.g. in contrast to integration of continuous functions.
- Analysing this construction in traditional computational analysis requires a way of representing the objects, a representation process that is itself of type 3.
- Of less interest, but still, they demonstrate that partial functionals of type 3 do appear in “nature" (see Platek's thesis for how they appear in “theory").

Ω_{C}

- When X is compact, let $\nu_{\mathrm{C}}(X)$ be the least element of X if $X \neq \emptyset$ and 0 if $X = \emptyset$.
- Ω_{C} and ν_{C} are computationally equivalent modulo $\exists^2$.
- The fact that an open subset O of $\mathbb{R}$ can be written as the union of a denumerable sequence of open intervals with rational end-points is, rightfully, considered to be trivial.

More on Ω_{C}

- Finding this set of rational intervals is actually an application of Π^1_1-comprehension with a type 2 parameter. This is a pretty strong comprehension principle.
- To "decide" if $(r, s) \subseteq O$, we can use Ω_{C}, the "procedures" are equivalent.
- Ω_{b} and Ω_{C} are closely related: the theorems and corollary about Ω_{b} two slides ago can be stated and proved for Ω_{C}.
- The conjecture is that Ω_{C} cannot be computed in Ω_{b} and $\exists^2$.

Classical hierarchies

- There are two well established hierarchies, one of the Borel sets and one of the Baire functions, based on iterations of basic set theoretic operations like taking the complement of a set, countable unions of sets and pointwise limits of functions.
- A natural question is when it is possible to select a way to generate an element in a particular level of one of these hierarchies without making a bow to the axiom of choice.
- Ω_{C} and $\exists^2$ does the job for open and closed subsets of $\mathbb{R}$ and for compact subsets of $\mathbb{N}^{\mathbb{N}}$.

Known facts

- There is no way to do this for $\mathbf{F}_\sigma$, for $\mathbf{G}_\delta$ or for Baire 2.
- Topologists know how to use transfinite recursion to construct representations of Baire 1 functions and of elements of $\mathbf{F}_\sigma \cap \mathbf{G}_\delta$.
- We have established that these constructions can be formalised as being computable in weak examples of Ω-functionals.

An example - Scattered sets

- A set $X \subseteq \mathbb{R}$ is *scattered* if every subset of X has an isolated point.
- Recall that $\Gamma_{\rm scat}$ is the class of scattered sets, and $\Omega_{\rm scat}$ be the corresponding quantifier.
- It is known that a set is scattered if and only if it is both countable and $\mathbf{G}_\delta$.
- $\Omega_{\rm scat} + \exists^2$ can compute a realisation of this from a scattered set X, using our functional Ξ.

Outline of proof

Let X be scattered, and let $\{B_n\}_{n\in\mathbb{N}}$ be an effective enumeration of all open intervals with rational endpoints.

- If C is closed, then Ω_{scat} and $\exists^2$ can decide from C when B_n contains exactly one element in $X \cap C$.
- We define a monotone function $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ as follows:
- Let $A \subseteq \mathbb{N}$ and $C = \mathbb{R} \setminus \bigcup_{i\in A} B_i$.
- Then $n \in F(A)$ if $B_n \cap C = \emptyset$ or if n is minimal such that $B_n \cap C \neq \emptyset$ and $B_n \cap X \cap C$ contains at most one element.

Outline continued

- F will be computable in $\Omega_{\rm scat}$ and $\exists^2$.
- $\mathbb{N}$ will be the only fixed point of F.
- F will generate a prewellordering of $\mathbb{N}$, and implicitly an increasing sequence $\{O_\alpha\}_{\alpha \leq \alpha_0}$ of open sets covering $\mathbb{R}$.
- For each $\alpha < \alpha_0$, $O_{\alpha+1} = O_\alpha \cup B_n$ for some n such that $B_n \setminus O_\alpha \neq \emptyset$ and contains at most one element from X.
- If there is one such $x \in X$, we enumerate it by n.
- In both cases, $O_{\alpha+1} \setminus X$ will be a pairwise disjoint union of $\mathbf{F}_\sigma$-sets (assuming this for O_α).
- It remains to prove that the generated prewellordering is computable in $\Omega_{\rm scat}$, $\exists^2$ and X.

Baire-able functionals

- Let $G : 2^{\mathbb{N}} \mapsto 2^{\mathbb{N}}$ be an arbitrary functional of type 2.
- We say that G is *Baire-able* if G is monotone, and $\mathbb{N}$ is the only fixed point of G.
- Let $\Xi(G) = \preceq_G$ be the associated prewellordering, defined for Baire-able G.
- We have

Theorem

Let G be Baire-able

a) $\Xi(G)$ *is computable in* $\exists^2$ *and* Ω_1*, uniformly in G.*
b) Ξ *is not computable in any functional H of type 2.*

- The proof of a) is quite easy, and the proof of b) uses standard techniques from HOC.

THANKS AGAIN MARTÍN

and

thank you all.