Martín's Interval in Scotland

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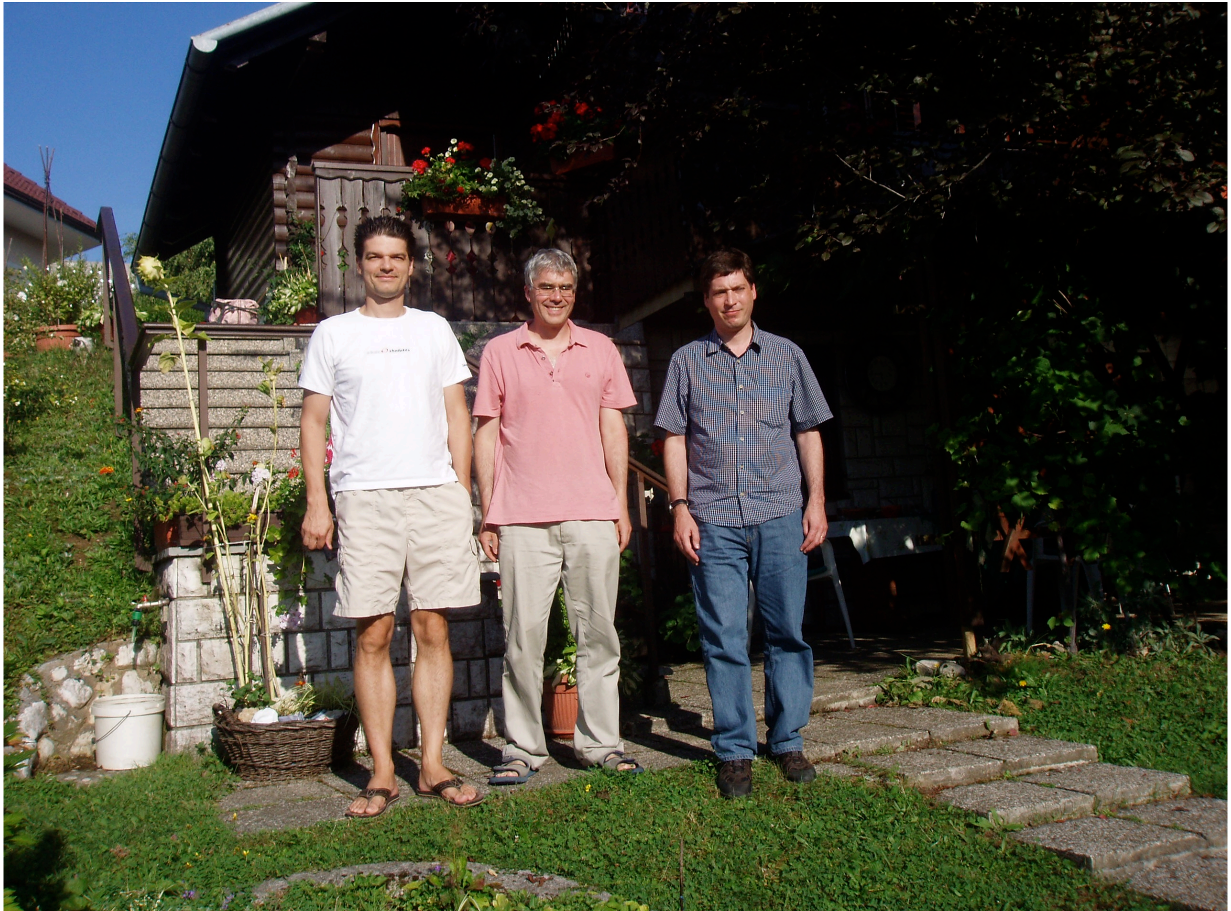


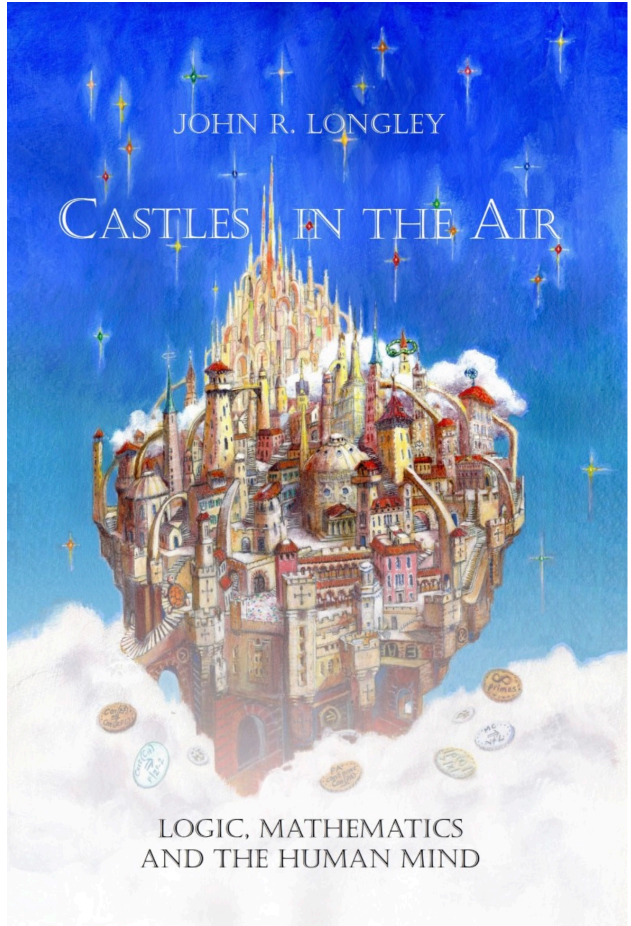
Martin's interval in Scotland: [1997, 2000]*

1997 – 1999: Edinburgh

1999 – 2000: St. Andrews

* A closed interval

cf. "Munro bagging in the effective topos"
LFCS "lab lunch" talk by John Longley, c. 1993


JOHN R. LONGLEY
CASTLES IN THE AIR
LOGIC, MATHEMATICS
AND THE HUMAN MIND

Martín's interval $[a,b]$

A free algebra I for $m: I^2 \to I$, $M: I^{\mathbb{N}} \to I$

$$m(x,x) = x$$

$$m(x,y) = m(y,x)$$

$$m(m(x,y), m(z,w)) = m(m(x,z), m(y,w))$$

A supermidpoint algebra

$$m(x,y) = M(x, y, y, y, y, \ldots)$$

$$M(x_1, x_2, x_3, \ldots) = m(x_1, M(x_2, x_3, x_4, \ldots))$$

$$\underset{i}{M}\,\underset{j}{M}\, x_{ij} = \underset{j}{M}\,\underset{i}{M}\, x_{ij}$$

over generators a, b.

(cancellation) $m(x,z) = m(y,z) \Rightarrow x = y$

<u>Theorem</u> The free cancellative supermidpoint algebra on 0,1 in <u>Set</u> is the closed interval $[0,1]$

<u>Theorem</u> The free cancellative supermidpoint algebra on 0,1 in <u>Top</u> is the closed interval $[0,1]$ with Euclidean topology

<u>Theorem</u> The free cancellative supermidpoint algebra on 0,1 in $\mathcal{E}$ is the smallest subset of the Dedekind interval $[0,1]$ containing rationals and closed under limits of Cauchy sequences (with modulus of convergence)

elementary topos with nno

Escardó & S.

- A universal characterization of the closed Euclidean interval.

  Proc. LICS 2001

- Euclidean interval objects in categories with finite products

  ArXiv 2025

A purely equational characterization?

Conjecture In **Set** $[0,1]$ is the free supermidpoint + flattening algebra on $0,1$.

Problem Characterize the Dedekind interval $[0,1]$ in an arbitrary elementary topos with nno.

Freyd has given one characterization, but it seems computationally uninformative.

A computational interpretation.

Escardó & S.

Abstract datatypes for real numbers in type theory

TLCA 2014

Extend System T with a type I and

$-1 : I \qquad 1 : I$

$m : I \to I \to I \qquad M : (N \to I) \to I$

$aff : I \to I \to I \to I$

System I

$$[\![I]\!] = [-1, 1]$$

$$\mathrm{aff} : I \to I \to I \to I$$

$$[\![\mathrm{aff}]\!]\ x\ y\ z = \frac{(1-z)x + (1+z)y}{2}$$

<u>Proposition</u> In System I, there is no closed term representing

$$\mathrm{dbl}(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{2} \\ 2x & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ -1 & \text{if } x \leq -\frac{1}{2} \end{cases}$$

System $\mathbb{I}\mathbb{I}$ $:=$ System $\mathbb{I}$ $+$ $\mathrm{dbl} : \mathbb{I} \to \mathbb{I}$

Theorem The functions $\mathbb{I}^n \to \mathbb{I}$ representable by closed terms of type $\mathbb{I}^n \to \mathbb{I}$ in System $\mathbb{I}\mathbb{I}$ are exactly the functions realizable by closed terms of system T of type $(\mathbb{N} \to \mathbb{N})^n \to (\mathbb{N} \to \mathbb{N})$ viewing real numbers as being represented in signed binary.

Extensional representability $=$ intensional representability.

Problem Does a similar theorem hold in a context in which only primitive recursive functions are available?

The full real line

$$\mathbb{R} := \varinjlim \quad \cdots \to [-\tfrac{1}{2}, \tfrac{1}{2}] \to [-1,1] \to [-2,2] \to [-4,4] \to \cdots$$

Represent real numbers using the type $Z \times I$ $\quad [\![Z]\!] = \mathbb{Z}$

Z an integer type

Can use $Z := N$
with an encoding

$$(m, x) \text{ represents } x \cdot 2^m$$

There is some intensionality

$$(m, x) \sim (m+1, x/2)$$

Defining functions on $Z \times I$

$$0 := (0,0)$$

$$1 := (0,1)$$

$$-(m,x) := (m,-x)$$

$$(m,x) + (n,y) := \left(m\left(\frac{x}{2^{\max(m,n)-m}} , \frac{y}{2^{\max(m,n)-n}} \right), \max(m,n)+1 \right)$$

$$(m,x) \cdot (n,y) := (x \cdot y,\ m+n)$$

Suppose $x_{(-)} : N \to (I \times Z)$ is <u>rapidly Cauchy</u> as witnessed by $d : N \to I$ satisfying

$$x_{n+1} - x_n \sim (d\,n\,,\ -(n+1))$$

this witnesses $|x_{n+1} - x_n| \leq 2^{-n}$

Then $\lim_i x_i$ is definable by $x_0 + (\underset{i}{M}\, d_i\,,\ 0)$

So we have a limit-finding program

$$\lim : (N \to (I \times Z)) \to (N \to I) \to (I \times Z)$$

$$\lambda x.\ \lambda d.\ x_0 + (M\,d\,,\ 0)$$

Theorem The System II - representable functions on $\mathbb{R}$ using $\mathbb{I} \times \mathbb{Z}$ coincide with the system $\mathbb{T}$ - representable functions on $\mathbb{R}$ using signed digits for the $\mathbb{I}$ component

Question Is every System II - representable function on $\mathbb{R}$ actually system I - representable?