

What Do Mapping Cylinders Classify?

In Celebration of Martín Escardó's 60th Birthday

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A parable inspired by Martín's mathematics

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Everything is very “civilised”: the sticky saps dry up, the buzzing insects zip away, and the last scurrying critters scatter to their holes.

Finally, all resistance muzzled, we can...show that the spectrum of a ring is a topological space.

But the thing about those little critters and buzzing bees and the sap that gets between your toes is—these entities were already an ecosystem with its own topology, geometry, arithmetic, computation—all intertwined together.

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I wish to be as quiet as I possibly can, so I may hear the chirping,

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vanish with the first application of force.

So thank you, Martín, for showing me by example how to listen.

I would like to share with you some things I have learned recently about partial map classifiers and their generalisations.

Some of this I learned from my collaborations with my students Leoni Pugh and Lingyuan Ye respectively.

Neutral synthetic topology in a lattice context

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Similar to *synthetic (domain theory, topology, algebraic geometry, etc.)* and *Abstract Stone Duality* (Taylor), but we do not assume axioms globally: our language is HoTT/UF .

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This is a contravariant adjunction

$$\mathcal{O}_\Sigma \dashv \mathrm{Spec}_\Sigma : \mathbf{Alg}_\Sigma^{\mathrm{op}} \rightarrow \mathbf{Set}$$

whose fixed points we call *quasi-coherent algebras* and *affine sets* respectively.

Many basic shapes arise as spectrums. For example, cubes and simplices:

$$\mathbb{I}^n = \text{Spec}_\Sigma \Sigma[x_1, \dots, x_n]$$

$$\Delta^n = \text{Spec}_\Sigma \Sigma[x_1, \dots, x_n] / x_i \geq x_{i+1}$$

Some important shapes are *not* spectrums, *e.g.* the generic inner horn Λ_1^2 obtained by glueing the interval onto itself end-to-end.

The interval \mathbb{I} is both a space (the spectrum of $\Sigma[x]$) and the underlying set of the algebra Σ .

1. Geometrical: interval is a figure shape for *homotopy*.
2. Logical: the inclusion $\{1\} \hookrightarrow \mathbb{I}$ determines a notion of *open subspace*.

Example: Phoa's principle

The following are equivalent for a Σ -algebra A :

1. The “generalised Phoa principle” (S. & Ye) holds for A .
2. For any element $\alpha \in A^{\mathbb{I}}$ we have $\alpha(i) = \alpha(0) \vee [i] \wedge \alpha(1)$.
3. The path space $A^{\mathbb{I}}$ classifies the canonical (lattice) order on A .
4. The polynomial Σ -algebra $A[x]$ is quasi-coherent.

Phoa's principle from *synthetic domain theory* states Σ itself satisfies the property above. Implications for synthetic higher categories too.

Other useful assumptions emerge by studying the monotone indicator function $[-] : \Sigma \rightarrow \mathbf{Prop}$ of the top element $1 \in \Sigma$.

1. Σ is *strict* when $[-] : \Sigma \rightarrow \mathbf{Prop}$ preserves the empty join.
2. Σ is *local* when $[-] : \Sigma \rightarrow \mathbf{Prop}$ preserves finite joins.
3. Σ is *conservative* when $[-] : \Sigma \rightarrow \mathbf{Prop}$ is an embedding.
4. Σ is *dominant* when it is conservative and the subuniverse $[-] : \Sigma \hookrightarrow \mathbf{Prop}$ is closed under internal sums.

NB: if every quotient of Σ by finitely many equations is quasi-coherent, then both Σ and Σ^{op} are dominant (S. & Ye; *cf.* Cherubini *et al.*).

Observations and open subspaces

Let X be a space. An *observation* of X is an element of the observational algebra $\mathcal{O}_\Sigma(X)$, i.e. a function $\varphi : X \rightarrow \mathbb{I}$.

Pulling back along $\{1\} \hookrightarrow \mathbb{I}$, we obtain the corresponding *open subspace*:

$$\begin{array}{ccc} X.\varphi & \longrightarrow & \{1\} \\ \downarrow & \text{cart.} & \downarrow \\ X & \xrightarrow{\varphi} & \mathbb{I} \end{array}$$

Open subspaces and open embeddings

We shall refer to an arrow of the form $U \hookrightarrow X$ arising from some observation as an *open embedding*. A *cloven open embedding* is defined to be an open embedding equipped with an observation whence it arises.

When Σ is conservative, all open embeddings are canonically cloven.

Closed subspaces and closed embedding

By duality, a *closed embedding* is an embedding $K \hookrightarrow X$ arising from an observation $\varphi \in \mathcal{O}_{\Sigma^{\text{op}}}(X)$ in the **dual lattice context**. In terms of Σ , this is a pullback of the inclusion $\{0\} \hookrightarrow \mathbb{I}$.

Open partial maps

An open partial map from X to Y is defined to be a *span*

$$\begin{array}{ccc} Y & \longleftarrow & U \\ & & \downarrow \\ & & X \end{array}$$

in which $U \hookrightarrow X$ is an open embedding.

Cloven open partial maps

A *cloven* open partial map from X to Y is a partial map equipped with a *cleaving* for its constituent open embedding:

$$\begin{array}{ccccc} Y & \longleftarrow & U & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow \\ & & X & \longrightarrow & \mathbb{I} \end{array} \quad \text{cart.}$$

The (cloven) open partial map classifier

The *partial product* of the generic open embedding $\{1\} \hookrightarrow \mathbb{I}$ with a space Y classifies cloven open partial maps into Y . Defining $L^\circ(Y) := \sum_{(i:\mathbb{I})} Y^{[i]}$, we consider the following open embedding:

$$\begin{array}{ccc} Y & \longrightarrow & \{1\} \\ \eta_Y^\circ \downarrow & \text{cart.} & \downarrow \\ L^\circ(Y) & \xrightarrow{\pi} & \mathbb{I} \end{array}$$

The (cloven) open partial map classifier (*cntd.*)

Every cloven open partial map from X to Y arises from a *unique* function $X \rightarrow L^\circ(Y)$ in the following way:

$$\begin{array}{ccccc}
 Y & \longleftarrow & U & \longrightarrow & \{1\} \\
 \downarrow & & \downarrow & & \downarrow \\
 & \text{cart.} & & \text{cart.} & \\
 L^\circ(Y) & \xleftarrow{\exists!} & X & \longrightarrow & \mathbb{I}
 \end{array}$$

Reminder: when Σ is conservative, we can forget all the cleavings.

The undefined partial element

When Σ is strict, *i.e.* we have $0 \neq 1$, the empty subset is open so we have:

$$\begin{array}{ccccc}
 X & \xleftarrow{!_X} & \mathbf{0} & \longrightarrow & \{1\} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{L}^\circ(X) & \xleftarrow{\text{und}_X} & \mathbf{1} & \xrightarrow{0} & \mathbb{I}
 \end{array}$$

cart. cart.

Here, $\text{und}_X : \mathbf{1} \hookrightarrow \mathbf{L}^\circ(X)$ is a *closed embedding*.

The “other” universal property of the partial map classifier

In ordinary domain/category/topos theory, the open partial map classifier $L^\circ(X)$ is *also* the **Sierpiński cone** X_\perp , which freely adds a bottom point.

$$\begin{array}{ccc}
 X & \xrightarrow{1_X} & X \\
 \downarrow & \nearrow & \downarrow \iota_X \\
 \mathbf{1} & \xrightarrow{\perp} & X_\perp
 \end{array}
 \qquad
 \begin{array}{ccccc}
 X & \xrightarrow{(0, X)} & \mathbb{I} \times X & \xleftarrow{(1, X)} & X \\
 \downarrow & & \downarrow \gamma_X & & \downarrow \iota_X \\
 \mathbf{1} & \xrightarrow{\perp} & X_\perp & &
 \end{array}$$

cocart.

NB: \perp is *closed* in X_\perp , and ι_X is an *open embedding* as soon as Σ is strict.

This is to say that the following evident square is cocartesian:

$$\begin{array}{ccc} X & \xrightarrow{(0, X)} & \mathbb{I} \times X \\ \downarrow & \text{cocart.} & \downarrow \text{glue}_X \\ \mathbf{1} & \xrightarrow{\text{und}_X} & L^\circ(X) \end{array}$$

where glue_X is the evident inclusion.

This *must not* hold synthetically for *generic* X .

A Brouwerian counterexample

If induced comparison map $X_{\perp} \rightarrow L^{\circ}(X)$ is an equivalence for *generic* X , then naturally each $[i]_{\perp} \rightarrow L^{\circ}[i]$ is an equivalence. But

$$[i]_{\perp} = \{j : \mathbb{I} \mid j = 0 \vee i = 1\}$$

$$L^{\circ}[i] = \{j : \mathbb{I} \mid j = 1 \rightarrow i = 1\}$$

and so for any $j \leq i$ we may conclude either $j = 0$ or $i = 1$. Considering $j \equiv i$ we therefore have $i = 0 \vee i = 1$, which makes all spaces codiscrete. When \mathbb{I} is strict, this makes the boundary inclusion $\mathbf{2} \hookrightarrow \mathbb{I}$ an equivalence.

Let the critters speak!

In domains/categories/toposes, $X_{\perp} \rightarrow \mathbf{L}^{\circ}(X)$ is an equivalence for *global* X , not for *generic* X . We could use a \flat -modality or switch to the Mitchell–Bénabou language to safely assert it only for global things.

But I am more interested in powering down the engines so that we can *hear* what the spaces are telling us in their own terms.

What the critters told me

1. It is not fair to compare X_{\perp} with $L^{\circ}(X)$ without fixing a universe of discourse, which (in synthetic mathematics) means an *accessible reflective subuniverse* in which to compute the pushout X_{\perp} .
2. “All” the comparison maps $X_{\perp} \rightarrow L^{\circ}(X)$ is far too large a left class to generate a reflective subuniverse, but **this localisation is nonetheless accessible and reflective**, and the presentation is *extremely* small.
3. For free we get a *limit-style* computation of *mapping cylinders* (Artin glueings) without higher inductive types.

The main result

Definition. A type X is called $[0]$ -connected when either of the following equivalent conditions hold:

1. The function space $X^{[0]}$ is contractible.
2. The injection $X \rightarrow [0] * X$ into the *join* is an equivalence.

Remark. The comparison map $X_{\perp} \rightarrow L^{\circ}(X)$ exists as soon as X is $[0]$ -connected.

Definition. A type C is called *Sierpiński complete* when the restriction map $(L^{\circ}(X) \rightarrow C) \rightarrow (X_{\perp} \rightarrow C)$ is an equivalence for generic $[0]$ -connected X .

The main result (*cntd.*)

Theorem. A type C is Sierpiński complete if and only if the *little* comparison map $(L^\circ[i] \rightarrow C) \rightarrow ([i]_\perp \rightarrow C)$ is an equivalence for all $i : \mathbb{I}$.

Remark. When Σ is conservative, the above is a slight *strengthening* of the Segal condition for synthetic $(\infty, 1)$ -categories.

Last year I proved a restricted version for 0-truncated C . Last month, I proved it unconditionally in Martin-Löf's intensional type theory with function extensionality, for an arbitrary bounded partial order Σ .

Constructing the Sierpiński extension

We start with an extension problem for C orthogonal to all $[i]_{\perp} \rightarrow L^{\circ}[i]$.

$$\begin{array}{ccc} X_{\perp} & \xrightarrow{f} & C \\ \downarrow & & \\ L^{\circ}X & & \end{array}$$

C is also orthogonal to the left maps below.

$$\begin{array}{ccc}
 \sum_{(i:\mathbb{I})} [i]_{\perp} \times X^{[i]} & \xrightarrow{\varepsilon_X^{\perp}} & X_{\perp} \xrightarrow{f} C \\
 \downarrow & & \downarrow \\
 \sum_{(i:\mathbb{I})} L^{\circ}[i] \times X^{[i]} & \xrightarrow{\varepsilon_X^{L^{\circ}}} & L^{\circ}X
 \end{array}$$

$$\varepsilon_X^{\perp} \langle i, u, x \rangle := x_{\perp}(u)$$

$$\varepsilon_X^{L^{\circ}} \langle i, u, x \rangle := L^{\circ}(x)(u)$$

So we compute the extension.

$$\begin{array}{ccccc}
 \sum_{(i:\mathbb{I})} [i]_{\perp} \times X^{[i]} & \xrightarrow{\varepsilon_X^{\perp}} & X_{\perp} & \xrightarrow{f} & C \\
 \downarrow & & \downarrow & & \uparrow \\
 \sum_{(i:\mathbb{I})} L^{\circ}[i] \times X^{[i]} & \xrightarrow{\varepsilon_X^{L^{\circ}}} & L^{\circ}X & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & & & C
 \end{array}$$

\hat{f}

Comment. It is easy to see, diagrammatically, that this construction gives a retraction to the restriction map $(\mathbf{L}^\circ[i] \rightarrow C) \rightarrow ([i]_\perp \rightarrow C)$.

It is *not* easy to see that it gives a *section* (i.e. that $\hat{f} \circ \delta$ actually extends f).

That this works fully coherently for untruncated spaces is my main new discovery over last year's paper with Leoni.

Some dependently typed geometry

Many geometrical constructions expressed globally are actually *sums* of simpler structures that can only be written down in type theory.

These type theoretic constructions are deeply suggestive...

Open and closed modalities

Given $P : \mathbf{Prop}$, we may form the *open modality*

$$\circlearrowleft_P X := X^P$$

and the *closed modality*

$$\bullet_P X := P * X.$$

Here, open/closed refers not to the topology induced by Σ but rather to the partitioning of the ambient higher topos \mathcal{E} into open/closed subtoposes. Nonetheless, there is a startling connection.

$L^\circ(X)$ vs. X_\perp in terms of modalities

The partial map classifier with respect to Σ is the *sum* of all the open modalities $\circ_{[i]}$, as we have $L^\circ(X) \equiv \sum_{(i:\mathbb{I})} \circ_{[i]} X$. Conversely, letting

$$\langle - \rangle : \mathbb{I} \rightarrow \mathbf{Prop}$$

be the indicator function for $0 \in \mathbb{I}$, we will see that the Sierpiński cone is the sum of all the *closed modalities* $\bullet_{\langle i \rangle}$, i.e. $X_\perp \cong \sum_{(i:\mathbb{I})} \bullet_{\langle i \rangle} X$.

A beautiful symmetry emerging from the failure of De Morgan's laws in constructive mathematics, which I learned from Reid Barton.

The Sierpiński cone is a sum of closed modalities

For generic $i : \mathbb{I}$, we compute the closed modality $\bullet_{\langle i \rangle} X$.

$$\begin{array}{ccc} \langle i \rangle \times X & \hookrightarrow & X \\ \downarrow & \text{cocart.} & \downarrow \\ \langle i \rangle & \longrightarrow & \bullet_{\langle i \rangle} X \end{array}$$

The Sierpiński cone is a sum of closed modalities

For generic $i : \mathbb{I}$, we compute the closed modality $\bullet_{\langle i \rangle} X$. Then we apply the left adjoint $\sum_{(i:\mathbb{I})} : \mathbf{Type}_{/(i:\mathbb{I})} \rightarrow \mathbf{Type}$, which preserves colimits.

$$\begin{array}{ccc} \sum_{(i:\mathbb{I})} \langle i \rangle \times X & \hookrightarrow & \sum_{(i:\mathbb{I})} X \\ \downarrow & \text{cocart.} & \downarrow \\ \sum_{(i:\mathbb{I})} \langle i \rangle & \longrightarrow & \sum_{(i:\mathbb{I})} \bullet_{\langle i \rangle} X \end{array}$$

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$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{I} \times X \\ \downarrow & \text{cocart.} & \downarrow \\ \mathbf{1} & \longrightarrow & \sum_{(i:\mathbb{I})} \bullet_{\langle i \rangle} X \end{array}$$

The open mapping cylinder

The Sierpiński cone X_{\perp} is a special case of the *open mapping cylinder*.

$$\begin{array}{ccc}
 E & \xrightarrow{1_E} & E \\
 \downarrow f & \nearrow & \downarrow \iota_f \\
 B & \xrightarrow{\perp_f} & M^{\circ}(f)
 \end{array}
 \qquad
 \begin{array}{ccccc}
 E & \xrightarrow{(0, E)} & \mathbb{I} \times E & \xleftarrow{(1, E)} & E \\
 \downarrow & \text{cocart.} & \downarrow \gamma_f & & \downarrow \iota_f \\
 B & \xrightarrow{\perp_f} & M^{\circ}(f) & &
 \end{array}$$

NB: $B \hookrightarrow M^{\circ}(f)$ is a closed embedding; when $f : E \rightarrow B$ is a $[0]$ -connected map, then $\iota_f : E \hookrightarrow M^{\circ}(f)$ is an open embedding.

The open mapping cylinder is a sum of Sierpiński cones

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$$\begin{array}{ccc} \mathbf{fib}_f(x) & \hookrightarrow & \mathbb{I} \times \mathbf{fib}_f(x) \\ \downarrow & \text{cocart.} & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{fib}_f(x)_\perp \end{array}$$

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$$\begin{array}{ccc} \sum_{(x:B)} \mathbf{fib}_f(x) & \hookrightarrow & \sum_{(x:B)} \mathbb{I} \times \mathbf{fib}_f(x) \\ \downarrow & \text{cocart.} & \downarrow \\ \sum_{(x:B)} \mathbf{1} & \longrightarrow & \sum_{(x:B)} \mathbf{fib}_f(x)_\perp \end{array}$$

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The open mapping cylinder $M^\circ(f)$ turns out to be the sum of all the Sierpiński cones of the *fibres* of f . We compute generically in $x : B$. Then we apply the left adjoint $\sum_{(x:B)} : \mathbf{Type}_{/(x:B)} \rightarrow \mathbf{Type}$. Simplify with Frobenius, and straightening/unstraightening.

$$\begin{array}{ccc} A & \hookrightarrow & \mathbb{I} \times A \\ \downarrow & & \downarrow \\ B & \longrightarrow & \sum_{(x:B)} \mathbf{fib}_f(x)_{\perp} \end{array} \quad \text{cocart.}$$

The open mapping cylinder of a *family*

From a type theoretic point of view, the function $f : E \rightarrow B$ could be viewed instead as a family $x : B \vdash E[-] : \mathbf{Type}$, whose open mapping cylinder we can define more directly.

$$\mathbf{M}^\circ_{(x:B)} E[x] := \sum_{(x:B)} E[x]_{\perp} \equiv \sum_{(x:B)} \sum_{(i:\mathbb{I})} \bullet_{\langle i \rangle} E[x]$$

(This is the open mapping cylinder of the fibration $\pi : \sum_{(x:B)} E[x] \rightarrow B$.)

The “logical” open mapping cylinder

All this suggests a “logical” or limit-style counterpart to the open mapping cylinder, using the partial map classifier:

$$\begin{aligned} \mathbf{M}^\circ_{(x:B)} E[x] &::= \sum_{(x:B)} E[x]_\perp \quad \cong \quad \sum_{(x:B)} \sum_{(i:\mathbb{I})} \bullet_{\langle i \rangle} E[x] \\ \mathbf{L}^\circ_{(x:B)} E[x] &::= \sum_{(x:B)} \mathbf{L}^\circ(E[x]) \quad \equiv \quad \sum_{(x:B)} \sum_{(i:\mathbb{I})} \circ_{[i]} E[x] \end{aligned}$$

A corollary for mapping cylinders

Let $x : B \vdash E[x]$ be a family of $[0]$ -connected types. Then we have an evident universal comparison map $M_{(x:B)}^\circ E[x] \rightarrow L_{(x:B)}^\circ E[x]$.

Corollary. Let C be Sierpinski complete. Then for any family $x : B \vdash E[x]$ of $[0]$ -connected types, C is orthogonal to this comparison map, *i.e.*

$$\left(L_{(x:B)}^\circ E[x] \rightarrow C \right) \rightarrow \left(M_{(x:B)}^\circ E[x] \rightarrow C \right)$$

is an equivalence.

What does the mapping cylinder classify?

Several years ago, I asked whether the Artin glueing of a geometric morphism classifies models of an interesting theory.

Our results imply an answer to the analogous question for **synthetic topology**: in suitable reflective subuniverses, mapping cylinders may be computed as sums of partial map classifiers. The subtlety is to ensure the reflective subuniverse is closed under mapping cylinders of *suitable*¹ families.

1. Leoni Pugh: automatic when Σ is local, dominant, and satisfies Phoa.
2. Otherwise we may adapt the theory of “well-completeness” from SDT.

¹Take care: the appropriate condition on the family is *not* fiberwise: we need the *fibration* $\sum_{(b:B)} E[x] \rightarrow B$ and the base B to be local.

thank you!

Future work

All these notions and results can be dualised to *closed embeddings* by considering the dual lattice Σ^{op} . We have Hyland's co-partial map classifier $\mathbf{L}^\bullet(X)$ and the inverted Sierpiński cone X^\top , and the closed mapping cylinders $\mathbf{M}^\bullet(f)$.

What about more general glueings? That would be good, but I am not yet sure how to proceed.