The cohomology of the natural numbers in cubical assemblies

Andrew W Swan

University of Ljubljana

December 18, 2025

Recall that the *first cohomology group with integer coefficients* of a type A, is defined as follows.

$$H^n(A;\mathbb{Z}) :\equiv \|A \to \mathbb{S}^1\|_0$$

Theorem

If A is projective, then $H^1(A;\mathbb{Z})$ is trivial.

Proof.

Fix $f : A \to K(G,1)$. For each $a : A$, $f(a) = \mathtt{base}$ is an inhabited set. Since A is projective, there exists a dependent function $p : \prod_{a:A} f(a) = \mathtt{base}$. p witnesses that f is equal to the identity of the group. □

Blass: Working over **ZF**, if A has trivial first cohomology for all abelian group coefficients, then it is projective.

In this way we can think of cohomology groups of sets as "measures" of the failure of the axiom of choice. In particular $H^1(\mathbb{N};\mathbb{Z})$ measures the failure of countable choice.

The definition of cohomology we gave before is internal to HoTT. Today we will mainly consider models of HoTT. In particular we consider models such as cubical sets that have an interval object $d_0, d_1 : 1 \rightrightarrows \mathbb{I}$ that can be used to define *paths* and *homotopies*.

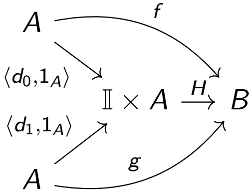
We will later restrict to cubical models with the following properties.

1. The underlying category is a presheaf topos.
2. The interval object $\mathbb{I}$ is representable.
3. "Cofibrations are finitary:" if Φ is the classifier for cofibrations, then every sieve belonging to Φ is finitely generated.

By carrying out the cubical set construction internally in assemblies, we obtain *cubical assemblies* (Uemura). Much of the construction can be carried out by simply working with cubical sets in a constructive meta theory.

Definition

Two maps $f, g : A \rightrightarrows B$ are *homotopic* if there is $H : \mathbb{I} \times A \to B$ fitting in the following commutative diagram.



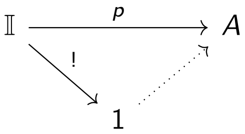
We can obtain a group in sets through the following construction.

Definition

Given a fibrant cubical assembly G satisfying the axioms of a group internally in the model and any cubical assembly B, we say the *B-externalisation of G* is the group $\mathrm{hom}(B, G)/{\sim}$, where $\sim$ is the homotopy equivalence relation.

Definition

A *path* in a cubical set A is a map $\mathbb{I} \to A$. A path $p : \mathbb{I} \to A$ is *degenerate* if it factors (necessarily uniquely) through 1:



Let P be a proposition (in our metatheory). Write $\mathbb{I}/P$ for the result of quotienting $\mathbb{I}$ by the constant cubical set on P.

Lemma

$\mathbb{I}/P$ is degenerate iff P is true.

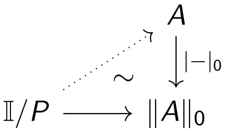
Lemma

A path $p : \mathbb{I} \to A$ factors (necessarily uniquely) through $\mathbb{I}/P$ iff P implies that p is degenerate.

Using the explicit description of HITs in cubical sets (Coquand-Huber-Mörtberg) and our technical conditions, we get the following lemmas. Key idea: It is decidable whether or not a path in the HIT uses any path constructors.

Lemma

We can lift every map $\mathbb{I}/P \to \|A\|_0$ to A up to homotopy, as illustrated below.



Lemma

For any cubical set A, and maps $f, g : \mathbb{I}/P \to A$, $|-|_0 \circ f$ and $|-|_0 \circ g$ are homotopic iff either P is false or $f = g$.

Define $\mathbb{N}_\infty$ to be the set of $\alpha : \mathbb{N} \to 2$ such that $\sum_{n:\mathbb{N}} \alpha(n) = 1$ has at most one element. We specialise the lemmas before setting $A := \mathbb{N} \to \mathbb{S}^1$ and P to be the proposition $\alpha = \infty$ for some $\alpha : \mathbb{N}_\infty$.
We consider the further variation where we replace $\mathbb{I}$ with the "non fibrant circle" $\mathbb{S}^1_0$ i.e. the coequalizer of the endpoint maps d_0, d_1. Paths $\mathbb{I} \to A$ factor (necessarily uniquely) through $\mathbb{I} \to \mathbb{S}^1_0$ iff the endpoints are strictly equal, i.e. if the path is a loop.

Theorem

We can lift maps $\mathbb{S}^1_0/(\alpha = \infty) \to H^1(\mathbb{N}, \mathbb{Z}) = \|\mathbb{N} \to \mathbb{S}^1\|_0$ to $\mathbb{N} \to \mathbb{S}^1$:

$$\begin{array}{ccc} & & \mathbb{N} \to \mathbb{S}^1 \\ & \overset{\sim}{\nearrow} & \big\downarrow |-|_0 \\ \mathbb{S}^1/(\alpha = \infty) & \longrightarrow & \|\mathbb{N} \to \mathbb{S}^1\|_0 \end{array}$$

Two maps $f, g : \mathbb{S}^1_0/(\alpha = \infty) \to (\mathbb{N} \to \mathbb{S}^1)$ are identified in $H^1(\mathbb{N}; \mathbb{Z})$ iff $f = g$ or $\alpha \neq \infty$.

Theorem

Let G be the subgroup of $\mathbb{Z}^{\mathbb{N}_\infty \times \mathbb{N}}$ consisting of $g : \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{Z}$ such that if $\alpha = \infty$, then $g(\alpha, n) = 0$ for all $n : \mathbb{N}$. Let K be the subgroup of G consisting of $g \in G$ such that for all α, either $\alpha \neq \infty$ or $g(\alpha, n) = 0$ for all $n : \mathbb{N}$.

$$\mathrm{hom}\Big(\sum_{\alpha:\mathbb{N}_\infty} \mathbb{S}^1/(\alpha = \infty), H^1(\mathbb{N};\mathbb{Z})\Big)/\sim \quad \cong \quad G/K$$

Corollary

If $H^1(\mathbb{N}; \mathbb{Z})$ is trivial in cubical sets, then **WLPO** *holds.*

Proof.

If $H^1(\mathbb{N}; \mathbb{Z})$ is trivial, then so is the externalisation G/K.
Consider $g : \mathbb{N}_\infty \times \mathbb{N} \to \mathbb{Z}$, defined by $g(\alpha, n) := \alpha(n)$. If $\alpha = \infty$, then $g(\alpha, n) = 0$ for all n, so $g \in G$. For g to lie in K, we would need that for all α, either $\alpha \neq \infty$ or $g(\alpha, n) = 0$ for all n. The latter implies $\alpha = \infty$, so we have $\alpha \neq \infty$ or $\alpha = \infty$ for all α. □

We can combine realizability with a related externalisation to get an ordinary group that has non trivial structure even with classical logic, including the following features.

Theorem

1. *torsion free*
2. *countable*
3. *infinite rank, i.e. has an infinite* $\mathbb{Z}$*-linearly independent subset.*
4. *contains a subgroup isomorphic to* $\mathbb{Q}$
5. *it has pure elements, i.e.* g *such that if* $mg = nh$ *for some* $m, n : \mathbb{Z}, h : G/K$*, then* $m|n$ *and* $h = \frac{n}{m}g$*.*

1. By showing the cohomology group of $\mathbb{N}$ is non trivial we get an informative proof that countable choice fails in cubical sets.
2. The key point is that propositional truncation in cubical sets is "too well behaved." When we construct a path in a truncation $\|A\|_{-1}$ we need to be able to decide whether or not any path constructors are needed. This results in families of elements of $H^1(\mathbb{N};\mathbb{Z})$ that are not trivial unless WLPO holds.
3. Using an externalisation via realizability, we can get an explicit group in classical mathematics, that is non trivial and has other interesting features.

Thanks for you attention!