Escardó's Patch

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What is the Patch construction?

The patch construction is a way of getting a compact Hausdorff locale from a stable locally compact locale:

$$\mathbf{StLocKLoc} \xrightarrow{\textit{Patch}} \mathbf{KHausLoc}$$

But what are stably locally compact locales? They are:

- continuous frames that are compact and have a meet-stable way below relation $\ll$.
- injective in **Loc** with respect to flat inclusions
- retracts of coherent locales
- 'reasonable' not necessarily Hausdorff compact spaces.

Compact Hausdorff (=compact regular) locales are stably locally compact: the *Patch* construction is the coreflection of the inclusion. *Also:*

$$\mathbf{StLocKLoc} \underset{\simeq}{\xrightarrow{\textit{Patch}}} \mathbf{KHausPos}$$

- can relate back to Priestley duality (restrict to coherent locales/spectral spaces)
- *Patch* construction involves taking the free Boolean algebra on the compact opens

Escardó's construction of Patch

Construction

[E01] For any stably locally compact locale, Y, $Patch^E(Y)$ is given by

$$\mathcal{O}Patch^E(Y) = \{j : \mathcal{O}Y \longrightarrow \mathcal{O}Y | j \text{ a nucleus}, j \text{ preserves directed joins}, \bigvee^{\uparrow}\}$$

Shown to me by Escardó in 1995/6 at Imperial College. Simple! No 'free Boolean algebra'.

My construction...

$$\mathcal{O}Patch(Y) = \textit{the frame generated by the images of the inclusions:}$$
$$\mathcal{O}Y \hookrightarrow idl(B_{\mathcal{O}Y}) \textit{ and } \Lambda\mathcal{O}Y \hookrightarrow idl(B_{\mathcal{O}Y}),$$

where $B_{\mathcal{O}Y}$ is the free boolean algebra on $\mathcal{O}Y$ keeping the distributive lattice structure fixed and Λ is taking all the Scott open filters.

...not so nice!

I did notice the following aspect: $\mathcal{O}Patch^E(Y) \subseteq \mathbf{PreFr}(\mathcal{O}Y, \mathcal{O}Y)$, but I could not see how we had such different constructions for the same thing.

Given X a compact Hausdorff locale I proved [T95] (really: Vickers showed me ...)

$$\mathit{CSub}(X \times X) \cong \mathbf{PreFr}(\mathcal{O}X, \mathcal{O}X)$$

so that closed relations in **KHausLoc** correspond to preframe homomorphisms and, further, relational composition goes to function composition.

Example

If $(X, \leq)$, we can form a subpreframe of $\mathcal{O}X$ by splitting the idempotent $\psi_{\leq}$, i.e. the preframe homomorphism corresponding to the relation $\leq$. It turns out that this subpreframe is the frame of opens of a stably locally compact locale, $\bar{X}$.

The problem of finding *Patch* was the problem of inverting this construction. With this example we had a localic account giving a spatial description of the inverse to *Patch* (take the topologically closed and upper closed sublocales), but could not extend to *Patch* itself.

If only we had paid more attention to our discrete motivation ...

Discrete poset motivation

Our interest in viewing stably locally compact locales in this way was by way of analogy with the discrete case: for any set L (=discrete locale) we have:

$$P(L \times L) \cong \mathbf{Sup}(PL, PL)$$

Example

If $(L, \leq)$ is a poset then $\mathcal{U}L$ (= upper closed subsets of L) is the splitting of the suplattice idempotent $\phi_{\leq} : PL \longrightarrow PL$. This splitting is the frame of opens of $Idl(L)$; the locale whose points are the ideals of L

So there is an analogy/duality between stably locally compact and algebraic dcpos. But recovering X from $\bar{X}$ by analogy seems hopeless as we no longer have the spatiality needed to extract L from $Idl(L)$. What we missed, only proved in 2011: **Proposition:** [T11] For any poset $(L, \leq)$, PL is the splitting of the idempotent

$$\mathcal{U}(L^{op} \times L) \longrightarrow \mathcal{U}(L^{op} \times L)$$

given by $R \mapsto \leq; (R \cap \Delta); \leq$.

Reconciling the Patches

So, in fact we can recover X from $\bar{X}$ as a splitting of the corresponding preframe idempotent:

$$\mathcal{O}\overline{X^{op} \times X} \longrightarrow \mathcal{O}\overline{X^{op} \times X}$$

defined by $R \mapsto \leq; (R \wedge \Delta); \leq$.

But, then, just as algebraic dcpos have a *-autonomous structure, we have the same for stably locally compact locales:

$$\begin{aligned} \mathcal{O}\overline{X^{op} \times X} &\cong \mathcal{O}\bar{X}^{op} \odot \mathcal{O}\bar{X} \\ &\cong \mathbf{PreFr}(\mathcal{O}\bar{X}, \Omega) \odot \mathcal{O}\bar{X} \\ &\cong \mathbf{PreFr}(\mathcal{O}\bar{X}, \mathcal{O}\bar{X}) \end{aligned}$$

so that it is actually 'obvious' why we have very different carrier sets for the patch construction.

Proposition: Let R be a closed relation on a compact Hausdorff localic poset $(X, \leq_X)$, such that: (i) R is contained in $\leq_X$, (ii) $R; R = R$ and, (iii) $R = \leq_X; R; \leq_X$. Then $R = \leq_X; (\Delta_X \wedge R); \leq_X$.
Proof: Because $R = \leq_X; R; \leq_X$ we have that $\psi_R : \mathcal{O}X \longrightarrow \mathcal{O}X$ restricts to a map $\bar{\psi}_R : \mathcal{O}\bar{X} \longrightarrow \mathcal{O}\bar{X}$. The other conditions then imply $Id \leq \bar{\psi}_R$ and $\bar{\psi}_R = \bar{\psi}_R \bar{\psi}_R$. So, $\bar{\psi}_R$ is in Escardó's patch. But, we have also given a description of the patch as the fixed points of $R \mapsto \leq_X; (\Delta_X \wedge R); \leq_X$; so, R must also enjoy this property. $\square$.

In other words R is determined by its restriction to the diagonal. This is not true on the discrete side: consider the strictly less than $<$ relation on the rationals $\mathbb{Q}$. This is an example of a regular statement that is not symmetric under discrete/compact Hausdorff duality.

Summary

- There should be a natural interest in stably locally compact locales: they are a key mathematical structure to understand in the context of locale theory.
- The patch construction naturally arises, both as a corflection back to compact Hausdorff locales, and as a way of representing compact Hausdorff localic posets.
- The patch construction also relates back naturally to other important dualities: e.g. Priestley duality.
- Escardó's patch was a particularly simple description, and contrasted to other constructions that involved having to construct free objects.
- It was not until later that we understood why so many apparently differing carrier sets existed into which the patch construction could be embedded.
- In fact the Patch construction (as an action on topologies) could be seen as the compact Hausdorff dual of a simple process of backing out the powerset of any poset from its set of left lower and right upper closed relations.
- The Escardó's patch construction then has an application: it provides a regular statement that is not dual under discrete/compact Hausdorff duality.

References

[E01] Escardó, M. *The regular-locally compact coreflection of a stably locally compact locale*, Journal of Pure and Applied Algebra, 157(1), 2001

[T95] Townsend, C. *Preframe techniques in constructive locale theory.* PhD, Imperial College (1996)

[T11] Townsend, C. *The patch construction is dual to algebraic dcpo representation.* Applied Categorical Structures 19 (1), 61-92, 2011.