Escardó-Simpson interval objects and point-free trigonometry [Vic25]

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$\mathbb{I} = [-1, 1]$ as Escardo-Simpson interval object [ES01]

If $A = \mathbb{R}^n$, $a_\pm \in A$, get unique affine $p\colon \mathbb{I} \to A$ with $\pm 1 \mapsto a_\pm$.

p maps $\mathbb{I}$ to convex hull of $\{-1, +1\}$, $\lambda \mapsto \frac{1-\lambda}{2} a_- + \frac{1+\lambda}{2} a_+$.
Idea: characterize $\mathbb{I}$ by maps like that, without already knowing about $\mathbb{R}$.

Structure needed on A (and $\mathbb{I}$)

- Midpoint algebra, midpoints $m(x, y)$ play role of $(x + y)/2$.
  p must preserve midpoints.
- Iterative

$$\forall \quad \begin{array}{c} X \xrightarrow{\ h\ } A \\ \circlearrowright t \end{array} \qquad \exists! \qquad \begin{array}{ccc} A \times X & \xrightarrow{A \times M} & A \times A \\ {\scriptstyle \langle h, t\rangle}\uparrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow[M]{} & A \end{array}$$

$\mathbb{I}$ is free iterative midpoint algebra on two elements ± 1.

Types "and" topology

- One approach: topology in type theory – eg synthetic topology
- By contrast: type theory in topology: bundles as dependent types.
- Spaces are generalized point-free (toposes, locales) [Vic07]
- *Ultra*minimalist foundations: geometric or arithmetic, no Π-types – regain access to points
- Point-free real analysis: see eg [NV22]

E-S Theorem for $\mathbb{I}$ as point-free space (locale) [Vic17]

$\mathbb{I}$ is iterative

- Use upper hyperspace (powerlocale) $\mathrm{P}_U\,\mathbb{I}$ of compact subspaces. $\mathbb{I}$ itself is *bottom* point of $\mathrm{P}_U\,\mathbb{I}$.
- Extend m to $m' \colon \mathbb{I} \times \mathrm{P}_U\,\mathbb{I} \to \mathrm{P}_U\,\mathbb{I}$.
- $\mathbb{I}$ is *metric space*, and $\mathrm{radius}(m'(a, K)) = 0.5 \times \mathrm{radius}(K)$.
- Define operation T on maps $f \colon X \to \mathrm{P}_U\,\mathbb{I}$:

$$
\begin{array}{ccc}
\mathbb{I} \times X & \xrightarrow{\mathbb{I} \times f} & \mathbb{I} \times \mathrm{P}_U\,\mathbb{I} \\
{\scriptstyle \langle h, t \rangle} \uparrow & & \downarrow {\scriptstyle m'} \\
X & \overset{T(f)}{\underset{f}{\rightrightarrows}} & \mathrm{P}_U\,\mathbb{I}
\end{array}
$$

- Let M be directed join of $\bot \sqsubseteq T(\bot) \sqsubseteq T^2(\bot) \sqsubseteq \ldots$.
  Each $M(x)$ has zero radius, so M factors through $\mathbb{I}$.

Iterativity and Cantor space 2^ω $\qquad (2 = \{-,+\})$

Iterative A equipped with two points, map $a\colon 2 \to \{a_-, a_+\}$ ...

$$\begin{array}{ccc} A \times 2^\omega & \xrightarrow{A \times M_{a_\pm}} & A \times A \\ {\scriptstyle \langle a \circ \mathrm{hd}, \mathrm{tl} \rangle} \uparrow & & \downarrow {\scriptstyle m} \\ 2^\omega & \xrightarrow[M_{a_\pm}]{} & A \end{array}$$

For $A = \mathbb{R}^n$,

$$M_{a_\pm}(s) = \left(\sum_{s_i = -} 2^{-i} \right) a_- + \left(\sum_{s_i = +} 2^{-i} \right) a_+ .$$

Do this with $\mathbb{I}$ and ± 1:
get $c\colon 2^\omega \to \mathbb{I}$.

$$c(s) = \sum_{i=1}^{\infty} \frac{s_i}{2^i}$$

$c\colon 2^\omega \to \mathbb{I}$ is a (point-free) surjection!

- Express it as a coequalizer.
- Factor $M_{a_\pm}$ through $N\colon \mathbb{I} \to A$.
- Show it's unique midpoint homomorphism taking ± 1 to $a_\pm$.

Trigonometry – measuring angles

Treat S^1 as unit circle in $\mathbb{C}$ under multiplication.
Wrap line round circle using group homomorphisms $\mathbb{R} \to S^1$, eg $\cos + i \sin$.

Idea: use midpoint homs $\mathbb{I} \to S^1$?

S^1 not a midpoint algebra ... but $S^1 - \{-1\}$ is.

- If $z = x + iy \in S^1$ then z has two square roots; but if $z \neq -1$ then exactly one of them has positive real part:

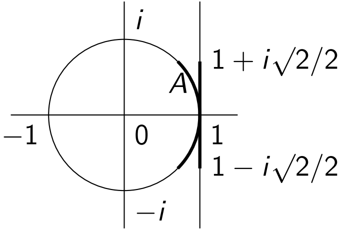
$$\sqrt{(x+iy)} = \frac{\sqrt{1+x}}{\sqrt{2}} + i\frac{y}{\sqrt{2}\sqrt{1+x}}.$$

- Define $m(z_1, z_2) = \sqrt{z_1} \cdot \sqrt{z_2}$.

Iterativity

Work with closed arcs in $S^1 - \{-1\}$ – iterativity proof with P_U uses compactness.
Can scale using $\sqrt{}$ and translate by multiplication, preserving midpoints, so one closed arc as good as another.

Let $A = \{z \in S^1 \mid \Re z \geq \sqrt{2}/2\}$.
$\Im$: $A \cong [-\sqrt{2}/2, \sqrt{2}/2]$,
doesn't preserve midpoints.
Makes A a metric space.



A is iterative

Same proof as for $\mathbb{I}$, except, for a suitable $\kappa < 1$ (approx 0.77),

$$\text{radius}(m'(a, K)) \leq \kappa \times \text{radius}(K)$$

Can extend that to other arcs.

Trigonometry

- Midpoint hom $\mathbb{I} \to S^1 - \{1\}$, $\pm 1 \mapsto \pm i$, gives $e^{ix\pi/2}$. Can extend domain, and adjust for other angular measures.
- $$\text{Define } \arcsin y := \int_0^y \frac{dt}{\sqrt{1-t^2}}, \quad \pi := 4 \arcsin\left(\frac{\sqrt{2}}{2}\right)$$

  Using inverse of arcsin we get midpoint homomorphisms to $S^1 - \{1\}$ and relate them to the ones already known. Then use Fundamental Theorem of Calculus [Vic23] to get derivatives of sin and cos.

Concluding thoughts

- Constructive account, closer to intuitions than power series.
- We can construct an $\mathbb{R}$-indexed family of homomorphisms $e^{iax} \colon \mathbb{R} \to S^1$. Are these all the homomorphisms? What can we say about Pontryagin duality? Or Fourier transforms?
- The midpoint homomorphisms from $\mathbb{I}$ are like geodesics. Can we extend the theory to cover curved midpoint algebras? (But one of the axioms is a *flatness* condition.)

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Bibliography II

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