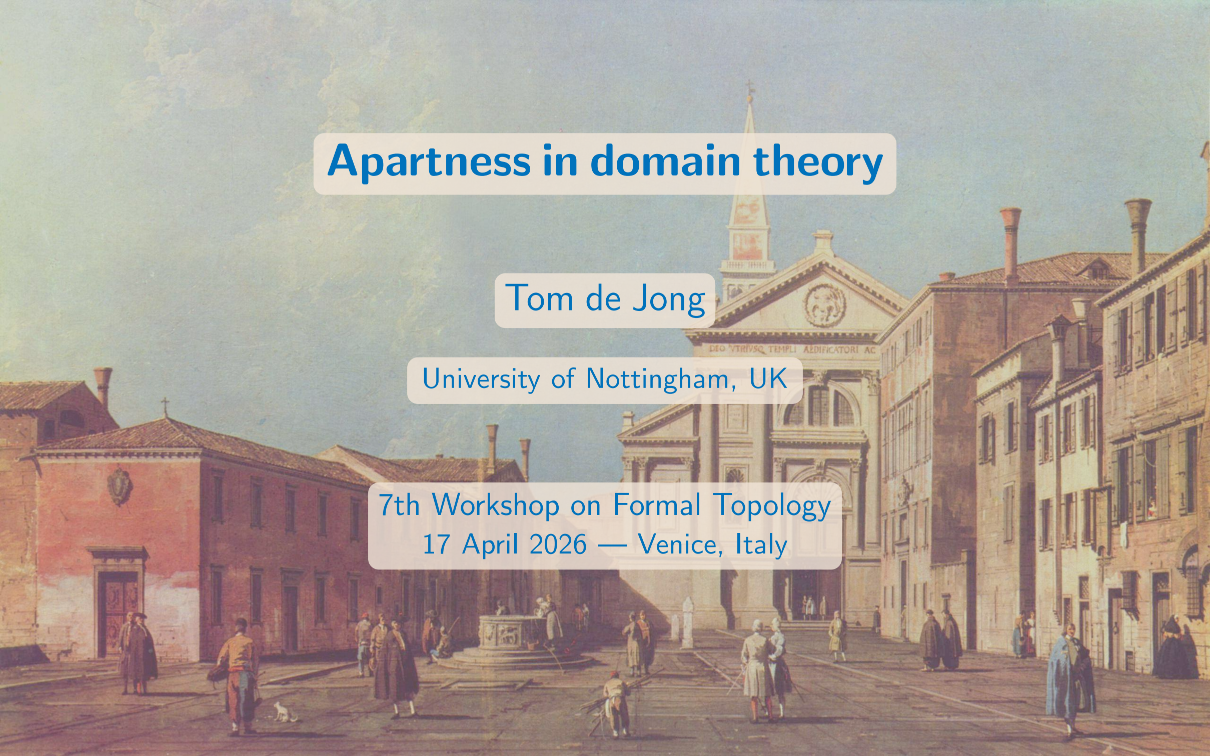

Apartness in domain theory

Tom de Jong

University of Nottingham, UK

7th Workshop on Formal Topology
17 April 2026 — Venice, Italy

Introduction

My Birmingham years: 2018–2022

- My PhD thesis
  Domain Theory in Constructive and Predicative Univalent Foundations
  School of Computer Science, University of Birmingham, 2023.

- Supervised by Martín Escardó.

- Co-supervised by Benedikt Ahrens.

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- I attended the 6$^{\text{th}}$ Workshop on Formal Topology as a 1$^{\text{st}}$ year PhD student.
  It's a privilege to be back 7 years later!

The origins of this talk

Two papers written in that period were not included in my thesis. They were

- (meant to be) foundation agnostic,
- constructive,
- not necessarily predicative.

Sharp Elements and Apartness in Domains
Proceedings 37th Conference on Mathematical Foundations of Programming Semantics (MFPS 2021)
Electronic Proceedings in Theoretical Computer Science 351, 2021.

Apartness, sharp elements, and the Scott topology of domains
Mathematical Structures in Computer Science 33.7, 2023.
Extended version of the above

In this talk, I will

- Give an overview of these papers.
- Illustrate (through examples!) how a constructive development highlights, and is informed by, the computational intuitions underlying domain theory.

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- Give an overview of these papers.
- Illustrate (through examples!) how a constructive development highlights, and is informed by, the computational intuitions underlying domain theory.
- Draw a few connections to recent work with Escardó on injective types.
- Discuss how Van der Weide & Frumin, Escardó, and Tosun built on my work since 2023.

Full references with links can be found at the end of these slides.

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- I work informally in a constructive, but impredicative theory, e.g. IZF, CIC (with unique choice?), higher-order topos logic, or HoTT with propositional resizing.

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I do *not* work with formal topologies. I hope this complements the work/talks by others in this workshop.

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- Applications in semantics of programming languages, higher-type computability, topology & locale theory.

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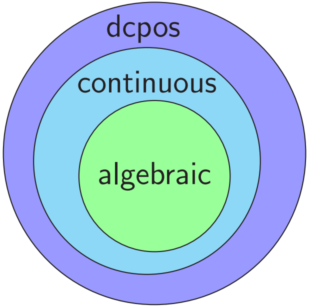
- All our examples will be continuous dcpos that have a basis whose elements can be used to approximate any other element.
- In the special case of algebraic dcpos every *"ideal"* element can be approximated using compact ("finitary") elements.
- Continuous dcpos are well behaved; algebraic dcpos even more and are good for building intuition.



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  This is the constructive counterpart to the *classical flat domain* $X_\perp := X + \{\perp\}$ with $\perp$ the least element and the rest ordered discretely.

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- A special case is the *Sierpiński domain* $\mathcal{S} :\equiv \mathcal{L}\{\star\} = \mathcal{P}\{\star\} = \Omega$.

  We informally identify a proposition p with the subset $\{\star \mid p\}$ so that $\perp :\equiv \emptyset$ and $\top :\equiv \{\star\}$ and p holds if and only if $\{\star \mid p\} = \top$.

  The compact elements of $\mathcal{S}$ are precisely $\perp$ and $\top$.

Algebraic domains embedding sequences

- For a set A, we can consider the set A^* of finite sequences of elements of A, ordered by prefix $\preceq$, and take its ideal completion $\mathcal{A} \coloneqq \mathrm{Idl}(A, \preceq) \subseteq \mathcal{P}(A^*)$.

  This is an algebraic dcpo (ordered by subset inclusion) whose compact elements are of the form $\{t \in A^* \mid s \preceq t\}$ for arbitrary $s \in A^*$.

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  The dcpo admits an injection

$$\begin{aligned} (\mathbb{N} \to A) &\hookrightarrow \mathcal{A} \\ \sigma &\mapsto \{s \in A^* \mid s \preceq \sigma\}. \end{aligned}$$

- By considering $A = \mathbf{2}$ and $A = \mathbb{N}$, we can respectively inject Cantor space $\mathbb{C}$ into the *Cantor domain* $\mathcal{C}$ and Baire space $\mathbb{B}$ into the *Baire domain* $\mathcal{B}$.

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  Taking the ideal completion of $(\mathbb{Q} \times_< \mathbb{Q}, \prec)$ we obtain the continuous dcpo $\mathcal{R}$ of *partial Dedekind reals*.

  A partial Dedekind real x is a real without the *locatedness* axiom, i.e.

  $$\forall (p, q \in \mathbb{Q}).\, p < q \implies (x < q) \vee (p < x)$$

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  We have an injection $\mathbb{R} \hookrightarrow \mathcal{R}$ given by $x \mapsto \{(p, q) \mid p < x < q\}$.

The intrinsic apartness of a domain

The Scott topology

- For a dcpo D, a *Scott open* U is a subset $U \subseteq D$ that is upwards-closed and *inaccessible by directed suprema*: if $\bigsqcup S \in U$, then $S \between U$ (i.e. $\exists s \in S \cap U$).

  Ex. In an algebraic dcpo D the basic Scott opens are precisely $\uparrow c := \{x \in D \mid c \sqsubseteq x\}$ for c compact.

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- Classically, a dcpo with the Scott topology satisfies T_0-separation: any two points with the same Scott open neighbourhoods must be equal.

  Constructively, this holds for continuous dcpos.

Telling points apart via Scott opens

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  Proof. Suppose $\neg\neg p$. Then $p \neq \bot$, so by assumption we a compact element $c \in \mathcal{S}$ with $p \in \uparrow c$ and $\bot \notin \uparrow c$. (The other case is impossible.) Since c is compact, it is either $\top$ or $\bot$. The latter is impossible, so $p \in \uparrow \top$ and hence $p = \top$. $\square$

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- This doesn't stop us. In fact, quite the opposite:

A positive reading yields a constructively valuable concept (cf. Sambin's *"dark side of the moon"*).

Giovanni Sambin. *Positive Topology. A New Practice in Constructive Mathematics*. 2025.

The intrinsic apartness of a dcpo

Def. We say that $x, y \in D$ are intrinsically apart, written $x \# y$ if we have a Scott open $U \subseteq D$ with either $x \in U$ and $y \notin U$, or $x \notin U$ and $y \in U$.

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The relation $\#$ is antireflexive and symmetric, but not necessarily cotransitive nor tight (more on this later):

$$x \mathrel{\#} y \implies (x \mathrel{\#} z) \vee (z \mathrel{\#} y) \qquad \text{(cotransitivity)},$$

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Prop. The injections $\mathbb{C} \hookrightarrow \mathcal{C}$, $\mathbb{B} \hookrightarrow \mathcal{B}$ and $\mathbb{R} \hookrightarrow \mathcal{R}$ all preserve and reflect apartness, where two sequences are apart if they differ at some index, and two reals x and y are apart if there exists $q \in \mathbb{Q}$ with $x < q < y$ or $y < q < x$.

Consequently, the elements in the images of these injections *do* satisfy cotransitivity and tightness.

The Scott and Bridges–Vîță topologies agree

Thm. Suppose that we have an algebraic dcpo such that *the order relation is decidable on compact elements*. Then the Bridges–Vîță apartness topology induced by the intrinsic apartness coincides with the Scott topology.

- In short, the Bridges–Vîță framework applies to domain theory.

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- In short, the Bridges–Vîță framework applies to domain theory.
- Note that the decidability condition holds in many examples of interest, e.g. $\mathcal{P}\,X$ for X with decidable equality as well as the Baire and Cantor domains.
- We also have a more general theorem for continuous dcpos with bases satisfying a variety of decidability conditions.

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Failure of cotransitivity and tightness

- Any set with a tight apartness has $\neg\neg$-stable equality, so if the Sierpiński domain $\mathcal{S}$ would have *any* tight apartness, then it would have $\neg\neg$-stable equality which is equivalent to excluded middle (in the form $\neg\neg p \implies p$).

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- If a dcpo has any cotransitive apartness $\sharp$ that is nontrivial in the sense that there are $x \sqsubseteq y$ with $x \sharp y$, then weak excluded middle ($\neg p \vee \neg\neg p$) holds.

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  For the intrinsic apartness on $\mathcal{S}$, cotransitivity yields full excluded middle.
- Special case: if a pointed dcpo has a cotransitive apartness $\sharp$ with elements $\bot \sharp x$, then weak excluded middle holds.
- The special case has a vast generalization: if an injective type has a cotrans. apartness with two points apart, then weak excluded middle holds.

TdJ and Martín Hötzel Escardó. *Examples and counterexamples of injective types*. 2026.

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$$p \implies s = x \quad \text{and} \quad \neg p \implies s = y,$$

so that taking contrapositives yields

$$x \neq s \implies \neg p \quad \text{and} \quad s \neq y \implies \neg\neg p.$$

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Moral of the story: completeness and discreteness are at odds with each other.

Digression: non-discreteness of injective types

- Recall that the carriers of pointed dcpos are examples of injective types (i.e. those types that admit extensions along type-theoretic embeddings).
- Thm. (with Escardó, 2026) An injective type has no nontrivial decidable property unless weak excluded middle holds.
- Apart from pointed dcpos, examples of injective types in univalent mathematics include type universes and various types of mathematical structure e.g. the type of groups, monoids, ordinals, etc.

Sharp elements

Sharp elements

Def. An element x of an algebraic dcpo D is *sharp* if $c \sqsubseteq x$ is decidable for every compact element $c \in D$.

Def. More generally, an element x of a dcpo D is *sharp* whenever $y \ll z$ implies $y \ll x$ or $z \not\sqsubseteq x$.

Prop. For a continuous dcpo:

- The intrinsic apartness is tight for sharp elements, so such elements are equal if they are not apart.
- The intrinsic apartness is cotransitive w.r.t. sharp elements: for every x and y and sharp element z, if $x \mathrel{\#} y$ then $(x \mathrel{\#} z) \vee (z \mathrel{\#} y)$.

Examples of sharp elements

Ex. In many examples of interest, all compact elements are sharp.

Ex. The elements in the images of the injections

$$\mathbb{B} \hookrightarrow \mathcal{B}, \quad \mathbb{C} \hookrightarrow \mathcal{C}, \quad \text{and} \quad \mathbb{R} \hookrightarrow \mathcal{R}$$

are all sharp. For the first two, this is because we can check finite prefixes of infinite sequences of booleans/naturals. For the reals, it is thanks to *locatedness*.

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Prop.

1. A pair $(x, y) \in D \times E$ is sharp if and only if both x and y are sharp.
2. An element $f : D \to E$ of the exponential of two bounded complete algebraic dcpos is sharp if and only if $f\ c$ is sharp for every compact element $c \in D$.

Sharpness in previous work

Def. (Spitters; Kawai) A subset V of a poset S is *located* if for every $s, t \in S$ with $s \ll t$, we have $t \in V$ or $s \notin V$.

Prop. An element x of a continuous dcpo is sharp if and only if the filter of Scott open neighbourhoods of x is located in the poset of Scott opens of D.

Bas Spitters. *Locatedness and overt sublocales*. 2010.

Tatsuji Kawai. *Geometric theories of patch and Lawson topologies*. 2017.

Strong maximality

Strong maximality in a **classical** context

- "Domain environments": embed a topological space into a domain, as its subspace of maximal elements e.g. $\mathbb{B} \hookrightarrow \mathcal{B}$, $\mathbb{C} \hookrightarrow \mathcal{C}$, and $\mathbb{R} \hookrightarrow \mathcal{R}$.
- Smyth published a paper (2006) on "constructively maximal" elements, a notion adapted from Martin-Löf (1970). Heckmann arrived at an equivalent notion in an unpublished manuscript (1998) and called it strong maximality.
- While the subspace of maximal elements of a domain may fail to be Hausdorff, the subspace of strongly maximal elements with the Scott topology is always Hausdorff and also regular (Smyth).

Michael (Mike) B. Smyth was Martín's PhD supervisor ☺

Strong maximality in a **classical** context

- "Domain environments": embed a topological space into a domain, as its subspace of maximal elements e.g. $\mathbb{B} \hookrightarrow \mathcal{B}$, $\mathbb{C} \hookrightarrow \mathcal{C}$, and $\mathbb{R} \hookrightarrow \mathcal{R}$.
- Smyth published a paper (2006) on "constructively maximal" elements, a notion adapted from Martin-Löf (1970). Heckmann arrived at an equivalent notion in an unpublished manuscript (1998) and called it strong maximality.
- While the subspace of maximal elements of a domain may fail to be Hausdorff, the subspace of strongly maximal elements with the Scott topology is always Hausdorff and also regular (Smyth).

- We adopt Heckmann's terminology and, based on a simplification of Smyth's definition, present a constructive treatment, connecting strong maximality to sharpness.

Michael (Mike) B. Smyth was Martín's PhD supervisor ☺

Strongly maximal elements

Def. An element x of an algebraic dcpo is *strongly maximal* if for every compact element c, either $c \sqsubseteq x$, or there is a compact element $d \sqsubseteq x$ with $\uparrow c \cap \uparrow d = \emptyset$.

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(We also have a more general definition for arbitrary dcpos.)

Prop. Every strongly maximal element x is maximal: if $x \sqsubseteq y$, then $x = y$.

Ex. The strongly maximal elements of the Baire, Cantor and partial Dedekind reals domains are precisely given by the image of Baire space $\mathbb{B} \hookrightarrow \mathcal{B}$, Cantor space $\mathbb{C} \hookrightarrow \mathcal{C}$ and the Dedekind reals $\mathbb{R} \hookrightarrow \mathcal{R}$.

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Proof. (For algebraic dcpos). Let x be strongly maximal and c compact. By assumption either $c \sqsubseteq x$, or we have a compact $d \sqsubseteq x$ with $\uparrow c \cap \uparrow d = \emptyset$. In the second case, we have $c \not\sqsubseteq x$ (else $x \in \uparrow c \cap \uparrow d = \emptyset$), so $c \sqsubseteq x$ is decidable. □

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Thm. In a continous dcpo, a point x is strongly maximal if and only if x is sharp and every Lawson* neighbourhood of x contains a Scott neighbourhood of x.

- (∗) This theorem requires a positive reformulation of the Lawson topology which, under excluded middle, is equivalent to the usual (classical) one.
- With excluded middle, all elements are sharp, and the condition on neighbourhoods is equivalent to the *Lawson condition* (the Scott and Lawson topologies coincide on the subset of maximal elements), so that we recover (classical) results by Smyth.

Sharpness and the points of the patch formal topology

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- Kawai:
  - Every geometric theory T presents a formal topology $\mathcal{S}_T$. The models of T are (equivalent to) formal points of $\mathcal{S}_T$.
  - For a stably loc. compact formal topology $\mathcal{S}$, its patch topology is presented by a geometric theory whose models are located formal points of $\mathcal{S}$.

    For the patch, see also Escardó (2001) and Coquand–Zhang (2023).
  - The Lawson topology of a continuous basic cover $\mathcal{S}$ is presented by a geometric theory whose models are the located subsets of $\mathcal{S}$.

    A continuous basic cover is the formal topology counterpart of a continuous lattice as introduced by Negri (1998) under a different name.

Tatsuji Kawai. *Geometric theories of patch and Lawson topologies*. 2017.

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Prop. For the Baire domain $\mathcal{B}$ this assertion is equivalent to *Markov's Principle*.
(MP) For every binary sequence σ, if $\neg\forall n \in \mathbb{N}.\, \sigma_n = 0$ then $\exists n \in \mathbb{N}.\, \sigma_n = 1$.

The subspace of strongly maximal elements (constructively)

Thm. The subspace of strongly maximal elements of a continuous dcpo with the Scott topology is Hausdorff *w.r.t. the intrinsic apartness*, i.e.

$$x \mathrel{\#} y \iff x \text{ and } y \text{ have disjoint Scott open neighbourhoods.}$$

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Thm. The restrictions of the injections $\mathbb{B} \hookrightarrow \mathcal{B}$, $\mathbb{C} \hookrightarrow \mathcal{C}$ and $\mathbb{R} \hookrightarrow \mathcal{R}$ to the strongly maximal elements are all homeomorphisms (where $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{R}$ are given their usual topologies).

How others have taken up the work

Domain-theoretic formalization of the reals

Van der Weide & Frumin:

- *Define* a real number to be a strongly maximal partial Dedekind real.

Niels van der Weide and Dan Frumin. *The Interval Domain in Homotopy Type Theory*. 2024.

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- *Define* a real number to be a strongly maximal partial Dedekind real.
- Show that the reals form a constructive ordered Archimedean field.
  The required apartness is just the intrinsic apartness (which is well behaved on strongly maximal elements).

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- Show that the reals form a constructive ordered Archimedean field.
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- Construct arithmetical operations on the reals by extending operations on rational intervals.

  Key: arithmetical operations preserve strong maximality, following Bauer & Taylor's abstract Stone duality (ASD).
- All formalized in Rocq using the UniMath library.

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Sharpness and conatural numbers (Escardó)

Def. Let $\mathbb{N}_\infty$ be the set of *conatural numbers*, i.e. (weakly) decreasing binary sequences.

$$\infty := 111\ldots \quad \in \mathbb{N}_\infty$$
$$\underline{n} := \underbrace{1\ldots1}_{n \text{ times}}000 \in \mathbb{N}_\infty \quad (n \in \mathbb{N})$$

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Thm. The image of ι is precisely the set of sharp elements of $\mathcal{L}\,\mathbb{N}$, so that

$$\mathbb{N}_\infty \cong \{x \in \mathcal{L}\,\mathbb{N} \mid x \text{ is sharp}\}.$$

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Sharpness and the points of patch locale, revisited

- Escardó's patch locale was revisited in predicative univalent foundations.

- Ayberk Tosun did his PhD on constructive and predicative locale theory in univalent foundations, and linked his formalized development to mine by formalizing the Scott locale Scott D of an algebraic dcpo D.

  The subset of compact elements is assumed to be suitably small.

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- Even better behaved are *Scott domains*, here defined as pointed, bounded complete algebraic dcpos such that *for every two compact elements it is decidable whether they have some upper bound*.

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Sharpness and the points of patch locale, continued

- Recall that a *point of a locale* X is a map $\mathbf{1} \to X$, i.e. a frame homomorphism $\mathcal{O}\,X \to \Omega$.
  Write $\mathsf{Pt}\,X$ for the collections of points of X.

- *Spectral points* can be characterized as those for which the subset given by the frame homomorphism has decidable membership for *compact* opens.
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- Writing $\#D$ for the sharp elements of a Scott domain D, Tosun showed:

$$\#D \;\cong\; \mathsf{Pt}_{\mathsf{spec}}(\mathsf{Scott}\,D) \;\cong\; \mathsf{Pt}_{\mathsf{spec}}(\mathsf{Patch}(\mathsf{Scott}\,D)) \;\cong\; \mathsf{Pt}(\mathsf{Patch}(\mathsf{Scott}\,D)).$$

Wrapping up

Summary

- Studied the intrinsic apartness on (continuous) dcpos that generalizes well known apartness relations on Baire space, Cantor space and the reals.
- Observed that the Bridges–Vîță approach to topology via apartness applies to domain theory.
- Identified the sharp elements as a subset of elements on which the apartness is well behaved (i.e. it is cotransitive and tight on sharp elements).
- Gave a constructive treatment of the subspace of strongly maximal elements which consists of sharp points and presents several topological spaces.
- Discussed how Van der Weide & Frumin, Escardó, and Tosun built on the above.

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- Lawson topology constructively?
  - Ayberk Tosun's talk: *Sharp Elements and the Points of the Patch Locale*.
  - Classically, Lawson $D \cong$ Patch(Scott D), but the RHS can fail to be spatial, e.g. for $D = \mathcal{P}\,\mathbb{N}$, the patch of its Scott locale is the Cantor locale, for which spatiality is equivalent to the Fan Theorem (Fourman & Grayson, 1982).
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Grazie!

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