Apartness for continuous dcpos

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Outline

Introduction

Overarching goal

Develop domain theory constructively (and predicatively) in Univalent Foundations.

Domain theory is a branch of order theory with applications in:

- semantics of programming languages;
- topology and algebra;
- higher-type computability.

See for instance [Str06; Sco72] for applications of domain theory to the semantics of programming languages and [AJ94; Sco72; Gie+80; Gie+03] for applications of domain theory in topology and algebra. See [LN15] for higher-type computability.

You can develop large parts of domain theory constructively and predicatively, e.g.

- Scott's model of PCF [Plo77; Sco93; Jon19];
- Scott's model D_∞ of the untyped λ-calculus [Sco72; JE21];
- continuous and algebraic dcpos, (abstract) bases and (rounded) ideal completion [AJ94; Gie+80; Gie+03; JE21].

In this talk we will:

- ignore the predicativity aspect of our work;
- not always give full definitions, but just sketch the main ideas.

Apartness: an introduction

- Constructively, we can distinguish between non-empty and inhabited; the latter being stronger.
- Similarly, constructively, we can distinguish between being unequal and being apart; the latter being stronger.
- Classically, these differences vanish.

Apartness: the basics

Definition (apartness, $\#$)

An apartness on a type X is a binary relation $\#$ such that for every $x, y : X$:

- $x \# y \to x \neq y$;
- $x \# y \to y \# x$ (*symmetry*).

Apartness: some examples

Example (Apartness on the reals $\mathbb{R}$)

Consider the Dedekind reals $\mathbb{R}$. We define two reals x and y to be apart if $\exists_{q:\mathbb{Q}}((x < q < y) \vee (y < q < x))$.

Example (Apartness on a metric space)

Let X be a space with a metric d. Then for $x, y : X$, we say that x and y are apart if their distance is strictly positive, i.e. $x \# y \iff d(x,y) > 0$.

Continuous dcpos: an introduction

Continuous dcpos are a “nice” class of posets with applications in:

- theoretical computer science;
- (pointfree) topology.

Many natural examples of dcpos are continuous.

Approximations of real numbers

- Consider pairs of rationals (p, q) with $p, q : \mathbb{Q}$ and $p < q$.
- We think of (p, q) as an approximation of a real number strictly between p and q.
- We order these pairs by

$$(p, q) \prec (r, s) \iff p < r < s < q.$$

  We say that (r, s) refines (p, q).
- We write $(\mathbb{Q} \times_{<} \mathbb{Q}, \prec)$ for this whole structure.

Question

How to model computations of real numbers?

Computations of real numbers

A computation of a real number will be modelled as a set c of approximations (p, q) such that:

- c is a downset (w.r.t. $\prec$):
  if $(p, q) \prec (r, s) \in c$, then $(p, q) \in c$;
  The intuition is: if we know that a real is between 2 and 3, then we also know that it is between 0 and 4.

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- c is inhabited;

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  The intuition is: if we know that a real is between 2 and 3, then we also know that it is between 0 and 4.
- c is inhabited;
- c is semidirected:
  if $(p_1, q_1), (p_2, q_2) \in c$, then there must be some $(p, q) \in c$ that refines both (p_1, q_1) and (p_2, q_2).
  The intuition is: we want to make *progress* and *exclude inconsistent* approximations

Ideals: computations of real numbers

Definition ((Rounded) ($\prec$-)ideals)

We call sets satisfying these properties rounded $\prec$-ideals, or just ideals for short. We order them by subset inclusion and get a poset: $\mathrm{Idl}(\mathbb{Q} \times_< \mathbb{Q}, \prec)$.

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- We have an embedding:

$$\begin{aligned} \iota : \mathbb{R} &\to \mathrm{Idl}(\mathbb{Q} \times_< \mathbb{Q}, \prec) \\ d &\mapsto \{(p, q) \in \mathbb{Q} \times_< \mathbb{Q} \mid p < d < q\} \end{aligned}$$

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$$\begin{aligned} \downarrow(-) : \mathbb{Q} \times_< \mathbb{Q} &\to \mathrm{Idl}(\mathbb{Q} \times_< \mathbb{Q}, \prec) \\ (p,q) &\mapsto \downarrow(p,q) :\equiv \{(r,s) \in \mathbb{Q} \times_< \mathbb{Q} \mid (r,s) \prec (p,q)\} \end{aligned}$$

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We think of $\downarrow(0,1)$ as a partial real number: a computation that does not converge to a real number.

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- All elements of the form $\iota(r)$ are maximal (w.r.t. $\subseteq$).
  This is because a Dedekind real d is located: if $p < q$, then $p < d$ or $d < q$.
- Classically, if an ideal x is maximal, then $x = \iota(d)$ for some Dedekind real number d.
- But constructively, this implication implies weak excluded middle.

Maximality, locatedness and weak excluded middle

Theorem

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- Consider the ideal

$$x :\equiv {\downarrow}(0,1) \cup \{(p,q) \mid p,q : \mathbb{Q}, p < 0 < q, \neg\neg\phi\}$$
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- Then x is maximal.

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- Then x is maximal.
- But if $x = \iota(d)$, then using locatedness of d, we can decide $\neg\neg\phi$.

□

Continuous dcpos: the basics

- In the example, $(\mathbb{Q} \times_< \mathbb{Q}, \prec)$ "generates" $\mathrm{Idl}(\mathbb{Q} \times_< \mathbb{Q}, \prec)$.

This key observation is well-known in the theory of continuous dcpos, see [Gie+03, p. 55 (end of the Remark just after Definition I-1.6)] and [AJ94, Proposition 2.2.10].

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- A continuous dcpo D generalizes and abstracts this:
  - D is a poset with order $\sqsubseteq$.
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  - Define a binary relation $\ll$ on D. If $x \ll y$, then we say that x is way below y.
    This generalizes $\prec$ and $\in$.

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Theorem (Key observation for us)

Let $x, y : D$. Then $x \sqsubseteq y \iff \prod_{b:B}(b \ll x \to b \ll y)$.
Intuitively, the right hand side says that every approximation of x is also an approximation of y.

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We wish to define apartness for continuous dcpos.
We define an intermediate notion, making use of:

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The notion of apartness in [Smy03] is very different from ours. Even in the presence of excluded middle, unequal points may fail to be "Smyth"-apart.

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Definition

We say that x is positively not below y, written $x \not\sqsubseteq y$ if $\exists_{b:B}(b \ll x) \times (b \not\ll y)$.
Intuitively, we have an approximation of x that is *not* an approximation of y.

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Note:

- $x \not\sqsubseteq y \to x \not\sqsubseteq y$;
- classically, $x \not\sqsubseteq y \to x \not\sqsubseteq y$.

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Apartness for continuous dcpos

- Since D is a poset: $x = y \iff x \sqsubseteq y \times y \sqsubseteq x$.
- This leads us to define an apartness by:

$$x \# y \iff x \not\sqsubseteq y \vee y \not\sqsubseteq x.$$

Apartness in our example

- Recall the embedding

$$\begin{aligned} \iota : \mathbb{R} &\to \mathrm{Idl}(\mathbb{Q} \times_< \mathbb{Q}, \prec) \\ d &\mapsto \{(p, q) \in \mathbb{Q} \times_< \mathbb{Q} \mid p < d < q\} \end{aligned}$$

- Recall that for two reals d and e we had:

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Thus our notion of apartness generalizes $\#_{\mathbb{R}}$.

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We have similar results with $\mathbf{2}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ replacing $\mathbb{R}$.

Connections with the Scott topology

- In classical topology, a topological space X is T_0-separated if for every two points $x, y : X$ we have:

$$x \neq y \iff \exists_{U \text{ open}}((x \in U \times y \notin U) \vee (y \in U \times x \notin U))$$

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- Dcpos have a natural topology: the Scott topology, which is T_0-separated, classically.
- Constructively, we have:

$$x \# y \iff \exists_{U \text{ Scott open}}((x \in U \times y \notin U) \vee (y \in U \times x \notin U))$$

Connections with the apartness topology

- There is a programme due to Bridges and Vîţă that develops topology constructively, taking apartness as primitive.
- Following Bridges–Vîţă, we can define the apartness topology on a continuous dcpo.

See the book [BV11] by Bridges and Vîţă.

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Theorem

Let D be a continuous dcpo with basis B. If the restriction of $\sqsubseteq$ to B is decidable, then the apartness topology on D and the Scott topology on D coincide.

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Example

Note that $\subseteq$ is *not* decidable on all of $\mathbb{R} \hookrightarrow \mathrm{Idl}(\mathbb{Q} \times_{<} \mathbb{Q}, \prec)$ (since equality of reals is not decidable), but it *is* decidable when restricted to elements of the form $\downarrow(p, q)$ (since $<$ is decidable on $\mathbb{Q}$).

See the book [BV11] by Bridges and Vîţă.

Conclusion

Introduced apartness for continuous dcpos.

- It generalizes some known notions of apartness (such as apartness on the reals).
- It is connected to the Scott topology and we get a constructive version of T_0-separation.
- Under a modest decidability condition, the apartness topology coincides with the Scott topology.

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