Acyclic types and epimorphisms in HoTT

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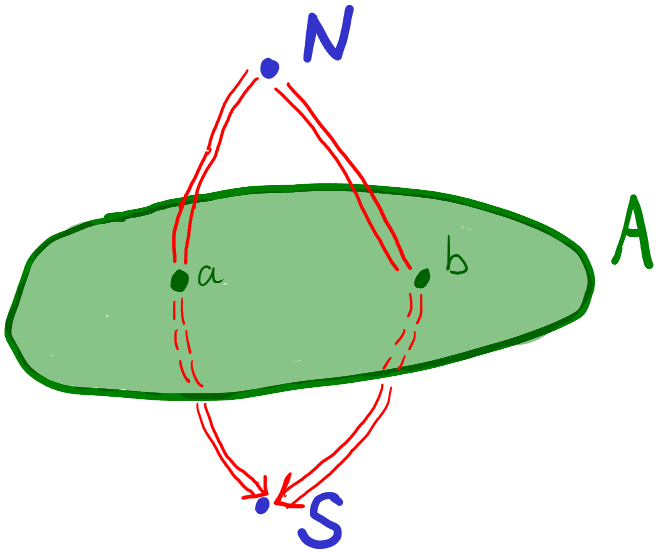

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\forall g"'] & B \arrow[dl, dashed, "\text{unique if it exists?}"] \\ X & \end{tikzcd}$$

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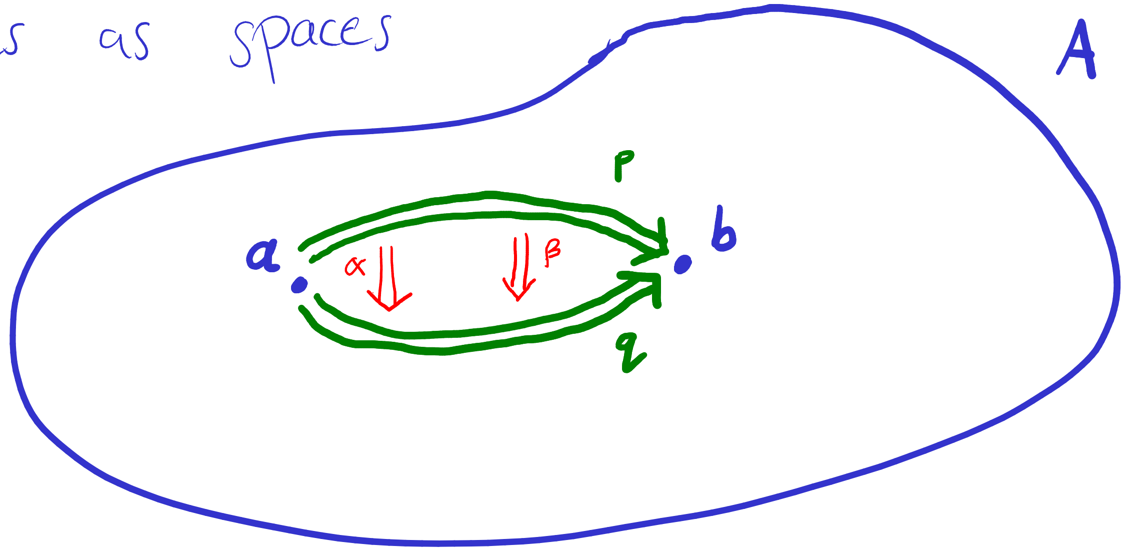
Outline

0. HoTT and preliminaries
1. Epimorphisms
2. Acyclic types and maps
3. Results and examples
4. Summary & future work

0. HoTT and preliminaries

"Types as spaces"



$a, b : A$

$p, q : a =_A b$

$\alpha, \beta : p =_{a =_A b} q$

$\vdots$

Def: A type is a set if its points can be equal in at most one way.

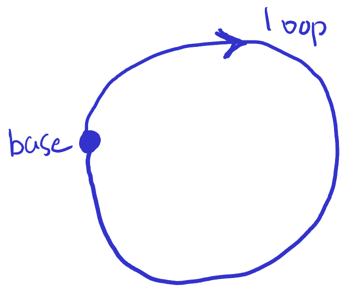
Example: $\mathbb{N}$, $\mathbb{0}$, $\mathbb{1}$, $\mathbb{2}$, $\mathbb{N} \to \mathbb{2}$, $\mathbb{N} \to \mathbb{N}$, etc.

We also call sets 0-types.

Higher types

The circle S^1

picture



higher inductive type

point constructor

$\text{base} : S^1$

path constructor

$\text{loop} : \text{base} =_{S^1} \text{base}$

fundamental group

Can prove in HoTT:

$\pi_1(S^1)$

$\simeq \text{base} =_{S^1} \text{base}$

$\simeq \mathbb{Z}$

Voevodsky's notion of equivalence

is a 1-type: its identity types are sets.

Iterating this definition we get the notion of a k-type for $k \geq 0$ (actually, $k \geq -2$).

1. Epimorphisms

A map $f : A \to B$ is an epi(morphism) in 1-category theory if for every $g, h : B \to C$ we have:

$$g \circ f = h \circ f \implies g = h.$$

That is, $(-) \circ f$ is injective.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\forall g"'] & B \arrow[dl, dashed, "\text{unique if it exists?}"] \\ X & \end{tikzcd} \iff f \text{ is an epi}$$

<u>Def</u>: A map $f : A \to B$ is an epi if $(-) \circ f$ is an embedding.

"HoTT version" of injection.

Def. (rep). A map $f: A \to B$ is an epi of types if for every type X the map

$$(B \to X) \xhookrightarrow{(-)\circ f} (A \to X)$$

is an embedding.

Def. A map $f: A \to B$ is an epi of k-types if for every k-type X the map

$$(B \to X) \xhookrightarrow{(-)\circ f} (A \to X)$$

is an embedding.

Fact: a map is an epi of 0-types/sets if and only if it is a surjection.

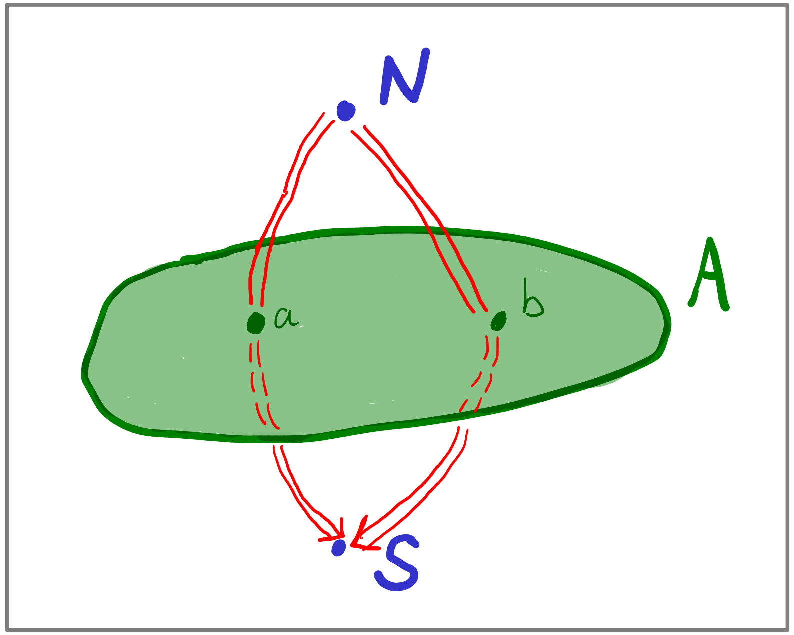
What happens for higher types?

To answer that we turn to <u>acyclic types</u>

2. Acyclic types

Def: The suspension ΣA of a type A is the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ \mathbb{1} \arrow[r] & \Sigma A \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$



Artist's rendition of ΣA

Def: A type A is acyclic if its suspension is contractible, i.e. $\Sigma A \simeq \mathbb{1}$.

Example:
- $\mathbb{1}$ is acyclic
- nontrivial examples in HoTT?

  in classical algebraic topology: ✓ (e.g. punctured Poincaré homology sphere)

Def. (rep.) A type A is acyclic if
if its suspension is contractible,
i.e. $\Sigma A \simeq \mathbb{1}$.

Def: A type A is k-acyclic if
if its suspension is k-connected
i.e. $\|\Sigma A\|_k \simeq \mathbb{1}$.

This denotes the k-truncation which is the universal way of making a type into a k-type.

Examples/facts:

- a type is 0-acyclic if and only if it is inhabited
- a type is 1-acyclic if and only if it is 0-connected, (e.g. $\mathbb{S}^1$, but not $\mathbb{N}$ or $\mathbb{2}$)
- every k-connected type is $(k+1)$-acyclic
  but not conversely, except when $k=0$.

More examples later...

Acyclic maps

<u>Def</u>: A map $f : A \to B$ is $(k\text{-})$acyclic if all of its fibres are $(k\text{-})$acyclic types.

$$\boxed{\begin{aligned} \mathrm{fib}_f : B &\to \mathrm{Type} \\ b &\mapsto \sum_{a:A} f(a) = b \end{aligned}}$$

3. Results and examples

Theorem: A map is an epi of (k-)types if and only if it is (k-)acyclic.

Micro version of the theorem

Proposition: A type A is $(k\text{-})$acyclic if and only if the map $A \to \mathbb{1}$ is an epi of $(k\text{-})$types.

Proof:

Standard fact: $f : A \to B$ is an epi iff the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "\mathrm{id}"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

is a pushout.

So, $A \to \mathbb{1}$ is epi iff

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathbb{1} \arrow[d] \\ \mathbb{1} \arrow[r] & \mathbb{1} \end{tikzcd}$$

is a pushout.

But this pushout defines the suspension of A, so this just says $\Sigma A \simeq \mathbb{1}$. $\square$

> **Theorem:** A map is an epi of (k-)types if and only if it is (k-)acyclic.

Proof of the ($\Rightarrow$)-direction:

Assume $f : A \to B$ is epic.
Then so is $\mathrm{fib}_f(b) \to \mathbb{1}$ for every $b : B$, because we have a pullback square

$$\begin{tikzcd}
\left(\sum_{a:A} f(a) = b\right) \equiv \mathrm{fib}_f(b) \arrow[r, "pr_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "\hat{b}"] & B
\end{tikzcd}$$

But by the micro version this means that $\mathrm{fib}_f(b)$ is acyclic for every $b : B$. □

Lemma: Epis are stable under pullback.

Proof: Suppose

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
Y \arrow[r] & B
\end{tikzcd}$$

f is epic.

Consider the commutative cube

$$\begin{tikzcd}
& X \arrow[dl] \arrow[d] \arrow[dr] & \\
Y \arrow[d] \arrow[dr] & A \arrow[dl] \arrow[dr] & Y \arrow[dl] \arrow[d] \\
B \arrow[dr] & Y \arrow[d] & B \arrow[dl] \\
& B &
\end{tikzcd}$$

Descent

> The sides are pullback squares and the bottom square is a pushout. Hence so is the top square. □

Acyclic groups

- Given a group G, we can construct a 0-connected 1-type BG with a point $pt : BG$ such that $pt =_{BG} pt$ and G are isomorphic as groups.
  We call BG the classifying type of G.

- Def: A group G is perfect if its abelianisation is the trivial group.

- **Theorem:** For a group G, the type BG is 2-acyclic if and only if G is perfect.

Acyclic sets

Theorem: The following are equivalent for a set A:
(i) A is 1-acyclic,
(ii) A is acyclic, and
(iii) A is contractible.

Proof: (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is clear. We prove (i) $\Rightarrow$ (iii).
Suppose A is 1-acyclic and let G be the free group on the set A with inclusion of generators $\eta : A \to G$.
Now, $A \to \mathbb{1}$ is an epi of 1-types, so

$$BG \xhookrightarrow{\text{const}} (A \to BG)$$

The embedding yields an equivalence on identity types

$$G \simeq (pt =_{BG} pt) \xrightarrow[\simeq]{\text{const}} (A \to (pt =_{BG} pt)) \simeq (A \to G)$$

So, $G \xrightarrow[\simeq]{\mathrm{const}} (A \to G)$.

Hence, $\eta : A \to G$ is constant.

Thus, $\eta(a) = \eta(b)$ for every $a, b : A$.

But η is injective, so all elements of A must be equal.

Finally, if A is 1-acyclic, then it is 0-acyclic and hence inhabited. Thus, $A \simeq \mathbb{1}$. $\square$

NB: The constructive fact that the inclusion of generators (from a set not necessarily having decidable equality) is an injection is due to Mines, Richman and Ruitenburg.

Martín has a nice Agda-formalised proof in HoTT developed with Bezem, Coquand and Dybjer.

4a. Summary

At higher types the notion of an epimorphism

- becomes quite strong,
- coincides with the notion of an acyclic map, and
- is interesting from the p.o.v. of synthetic homotopy theory.

4b. Future work

- Construct a nontrivial acyclic type.
- Do the acyclic maps form an accessible modality? (Classically they do.)
- How much can we do without needing Whitehead's Principle?

  $\boxed{\text{Every } \infty\text{-connected type is contractible.}}$

- Quillen's + construction
- Applications (where surjectivity doesn't cut it)

...

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