Predicative Aspects of Order Theory in Univalent Foundations

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Outline

Introduction

- Use Univalent Foundations (UF) (a.k.a. Homotopy Type Theory (HoTT)) to do constructive mathematics and formalize it in a proof assistant.

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- Predicative UF is UF without propositional resizing axioms.

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- Predicative UF is UF without propositional resizing axioms.

In this talk we show that some things cannot be done in predicative UF.

Main result in this talk

Theorem (crude formulation)

Various posets, such as sup-lattices, can only be small in impredicative UF, unless they are trivial.

An (impredicative) theorem by Freyd, for comparison

A category with small (co)limits is small if and only if it is a poset.

Impredicativity in UF

Definition

A *proposition* is a type with at most one element.

Definition

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Definition

A type $X : \mathcal{U}^{+}$ is *small* if we have $Y : \mathcal{U}$ with $Y \simeq X$.

Definition

The resizing axiom $\Omega\text{-Resizing}_{\mathcal{U}}$ asserts that $\Omega_{\mathcal{U}}$ is small.

Weak impredicativity in UF

Definition

A proposition P is *$\neg\neg$-stable* if $\neg\neg P$ implies P.

Definition

The type $\Omega_{\mathcal{U}}^{\neg\neg}$ is the type of all $\neg\neg$-stable propositions in a universe $\mathcal{U}$.

Definition

The resizing axiom $\Omega_{\neg\neg}\text{-Resizing}_{\mathcal{U}}$ asserts that $\Omega_{\mathcal{U}}^{\neg\neg}$ is small.

Excluded middle implies impredicativity

Definition

Excluded middle holds in a universe $\mathcal{U}$ if every proposition in $\mathcal{U}$ is either inhabited or empty.

Proposition

Excluded middle in $\mathcal{U}$ implies $\Omega\text{-Resizing}_{\mathcal{U}}$.

Proof.

With excluded middle in $\mathcal{U}$, we have $\Omega_{\mathcal{U}} \simeq \mathbf{1} + \mathbf{1}$. □

Definition

Weak excluded middle holds in a universe $\mathcal{U}$ if for every proposition P in $\mathcal{U}$ either $\neg\neg P$ holds or $\neg P$ does.

Proposition

Weak excluded middle in $\mathcal{U}$ implies $\Omega_{\neg\neg}\text{-Resizing}_{\mathcal{U}}$.

$\delta_{\mathcal{U}}$-completeness

Definition

A poset $(X, \sqsubseteq)$ is *$\delta_{\mathcal{U}}$-complete* if for every proposition $P : \mathcal{U}$ and elements $x \sqsubseteq y$, the family

$$\begin{aligned} \delta_{x,y,P} : \mathbf{1} + P \to X \\ \mathrm{inl}(\star) \mapsto x; \\ \mathrm{inr}(p) \mapsto y; \end{aligned}$$

has a supremum $\bigvee \delta_{x,y,P}$ in X.

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- With excluded middle in $\mathcal{U}$, every poset is $\delta_{\mathcal{U}}$-complete.
- If $x \neq y$, then $\bigvee \delta_{x,y,P} = x \iff \neg P$, while $\bigvee \delta_{x,y,P} = y \Rightarrow \neg\neg P$.

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- If the two-element poset with $0 \sqsubseteq 1$ is $\delta_{\mathcal{U}}$-complete, then weak excluded middle holds in $\mathcal{U}$.

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- If $x \neq y$, then $\bigvee \delta_{x,y,P} = x \iff \neg P$, while $\bigvee \delta_{x,y,P} = y \Rightarrow \neg\neg P$.
- If the two-element poset with $0 \sqsubseteq 1$ is $\delta_{\mathcal{U}}$-complete, then weak excluded middle holds in $\mathcal{U}$.
- $\mathcal{U}$-sup-lattices, i.e. posets with all $\mathcal{U}$-suprema, are $\delta_{\mathcal{U}}$-complete, as are $\mathcal{U}$-dcpos and $\mathcal{U}$-bounded complete posets.

Nontriviality and positivity

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- For $\delta_{\mathcal{U}}$-complete posets we can do better.

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Definition

An element x of a $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ is *strictly below* another element y if $x \sqsubseteq y$ and, moreover, for every $z \sqsupseteq y$ and every proposition $P : \mathcal{U}$, the equality $z = \bigvee \delta_{x,z,P}$ implies P.
We denote this by $x \sqsubset y$.

Definition

A $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ is *positive* if we have $x, y : X$ with $x \sqsubset y$.

Prime examples of $\delta_{\mathcal{U}}$-complete posets

Example

The powerset $\mathcal{P}(X) :\equiv X \to \Omega_{\mathcal{U}}$ is $\mathcal{U}$-sup-lattice.

- The pair $(\emptyset, A)$ witnesses nontriviality of $\mathcal{P}(X)$ if and only if A is a nonempty subset of X.
- The pair $(\emptyset, A)$ witnesses positivity of $\mathcal{P}(X)$ if and only if A is an inhabited subset of X.

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Example

The type $\Omega_{\mathcal{U}}$ is a $\mathcal{U}$-sup-lattice ordered by implication and with suprema given by existential quantification.

- The pair $(\mathbf{0}, P)$ witnesses nontriviality of $\Omega_{\mathcal{U}}$ if and only if $\neg\neg P$ holds.
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Example

The type $\Omega_{\mathcal{U}}^{\neg\neg}$ is a sub $\mathcal{U}$-sup-lattice of $\Omega_{\mathcal{U}}$.

Remark on being strictly below

Recall that we defined $x \sqsubset y$ as $x \sqsubseteq y$ and for every $z \sqsupseteq y$ and every proposition $P : \mathcal{U}$, the equality $z = \bigvee \delta_{x,z,P}$ implies P.

Remark on being strictly below

Recall that we defined $x \sqsubset y$ as $x \sqsubseteq y$ and for every $z \sqsupseteq y$ and every proposition $P : \mathcal{U}$, the equality $z = \bigvee \delta_{x,z,P}$ implies P.

- Why do we need for every $z \sqsupseteq y$?
- Why not just $y = \bigvee \delta_{x,y,P}$ implies P?

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- Why do we need for every $z \sqsupseteq y$?
- Why not just $y = \bigvee \delta_{x,y,P}$ implies P?
- Our stronger definition is used to prove:

Lemma

- *If $x \sqsubset y \sqsubseteq z$, then $x \sqsubset z$.*
- *If $x \sqsubseteq y \sqsubset z$, then $x \sqsubset z$.*

(Local) smallness

Definition

A $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ is *locally small* if we have $\sqsubseteq_{\mathcal{U}} : X \to X \to \Omega_{\mathcal{U}}$ such that $x \sqsubseteq_{\mathcal{U}} y \iff x \sqsubseteq y$ for every $x, y : X$.

Example

All the examples on the previous slide are locally small.

(Local) smallness

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Example

All the examples on the previous slide are locally small.

Definition

A $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ is *small* if it is locally small and its carrier X is small, i.e. we have a type in $\mathcal{U}$ equivalent to X.

Main theorems

Theorem

There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\neg\neg}\text{-Resizing}_{\mathcal{U}}$ holds.

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega\text{-Resizing}_{\mathcal{U}}$ holds.

One can replace "$\delta_{\mathcal{U}}$-complete poset" by "$\mathcal{U}$-sup-lattice" in both theorems.

Additional theorems

Theorem

There is a locally small nontrivial $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if weak excluded middle in $\mathcal{U}$ holds.

Theorem

There is a locally small positive $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if excluded middle in $\mathcal{U}$ holds.

Size matters

Definition

- A type X *has size* $\mathcal{U}$ if we have $Y : \mathcal{U}$ with $Y \simeq X$.
- A map $f : X \to Y$ *has size* $\mathcal{U}$ if all of its fibres do.

Lemma

If $f : X \to Y$ and Y have size $\mathcal{U}$, then so does X.

Proof.

The map

$$\begin{aligned} X &\to \sum_{y:Y} \mathrm{fib}_f(y) \equiv \sum_{y:Y} \sum_{x:X} f(x) = y \\ x &\mapsto (f(x), x, \mathrm{refl}) \end{aligned}$$

is an equivalence. □

Section-embeddings

Definition

A *section-embedding* is a section that is moreover an embedding.

Aren't all sections embeddings?

No, unless all types are sets.
But all sections to sets are embeddings.

Section-embeddings and size

Lemma

If $s : X \to Y$ is a section-embedding and Y has size $\mathcal{U}$, then so does s.

Proof.

Check that $\mathrm{fib}_s(y) \simeq \|s(r(y)) = y\|$ for every $y : Y$, where r is the retraction witnessing that s is a section. □

Corollary

If $s : X \to Y$ is a section-embedding and Y has size $\mathcal{U}$, then so does X.

Fundamental retract lemmas

Definition

For a $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ with points $x \sqsubseteq y$, we define

$$\begin{aligned}\Delta_{x,y} : \Omega_{\mathcal{U}} &\to X \\ P &\mapsto \bigvee \delta_{x,y,P}\end{aligned}$$

Lemma

A locally small $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ with points $x \sqsubseteq y$ is nontrivial if and only if the composite $\Omega_{\mathcal{U}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{U}} \xrightarrow{\Delta_{x,y}} X$ is a section.

Lemma

A locally small $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ with points $x \sqsubseteq y$ is positive if and only if for every $z \sqsupseteq y$, the map $\Omega_{\mathcal{U}} \xrightarrow{\Delta_{x,z}} X$ is a section.

Proof of the second lemma

Lemma

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Proof.

($\Rightarrow$): Let $r \sqsupseteq z$ and define $r_z : X \to \Omega_{\mathcal{U}}$ by $r_z(w) \mapsto z \sqsubseteq_{\mathcal{U}} w$. If $P : \Omega_{\mathcal{U}}$, then $r(\Delta_{x,z}(P)) \equiv r(\bigvee \delta_{x,z,P}) \iff (z = \bigvee \delta_{x,z,P}) \iff P$, by positivity.

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($\Leftarrow$): Let $z \sqsupseteq y$ and $P : \mathcal{U}$ a proposition. We must show that $z = \bigvee \delta_{x,z,P}$ implies P. Let $r_z : X \to \Omega_{\mathcal{U}}$ be a retraction witnessing that $\Delta_{x,z}$ is a section. Now if $z = \bigvee \delta_{x,z,P}$, then

$$\mathbf{1} = r_z(\Delta_{x,z}(\mathbf{1})) = r_z(z) = r_z(\Delta_{x,z}(P)) = P,$$

so P must hold. □

Main theorems

Theorem

There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\neg\neg}\text{-Resizing}_{\mathcal{U}}$ *holds.*

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega\text{-Resizing}_{\mathcal{U}}$ *holds.*

Additional results

Lemma

Types with decidable equality are closed under retracts.

Proof.

If $s : X \to Y$ is a section with retraction $r : Y \to X$ and Y has decidable equality, then we can decide $x_1 = x_2$ for every $x_1, x_2 : X$ by deciding $s(x_1) = s(x_2)$. □

Theorem

There is a locally small nontrivial $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if weak excluded middle in $\mathcal{U}$ holds.

Theorem

There is a locally small positive $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if excluded middle in $\mathcal{U}$ holds.

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Theorem

If $\mathcal{U}$ *and* $\mathcal{U}^+$ *are univalent, then* X *has-size* $\mathcal{U}$ *is a proposition for every* $X : \mathcal{U}^+$.

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Theorem

Suppose that $\mathcal{U}$ and $\mathcal{U}^+$ are univalent. There is a small $\delta_{\mathcal{U}}$-complete poset that is nontrivial in an unspecified way if and only if $\Omega_{\neg\neg}\text{-Resizing}_{\mathcal{U}}$ holds.

Theorem

Suppose that $\mathcal{U}$ and $\mathcal{U}^+$ are univalent. There is a small $\delta_{\mathcal{U}}$-complete poset that is positive in an unspecified way if and only if $\Omega\text{-Resizing}_{\mathcal{U}}$ holds.

Conclusion

Take-home message

- Nontrivial sup-lattices (dcpos, bounded-complete posets, etc.) can only be small in the presence of (weak) impredicativity.
- Nontrivial locally small sup-lattices (dcpos, bounded-complete posets, etc.) can only have decidable equality in the presence of (weak) excluded middle.

Preprint

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