Domain Theory in Constructive and Predicative Univalent Foundations

Tom de Jong[†] Martín Hötzel Escardó[†]

[†]University of Birmingham, United Kingdom

29th EACSL Annual Conference on Computer Science Logic
CSL 2021
Ljubljana, Slovenia

27 January 2021

UNIVERSITY OF BIRMINGHAM

Aim and motivation

Develop domain theory, but constructively and predicatively in univalent foundations.

Aim and motivation

Develop domain theory, but constructively and predicatively in univalent foundations.

Why domain theory?

- Classical topic in theoretical computer science
- Applications in:
  - semantics of programming languages;
  - topology;
  - higher-type computability.

Aim and motivation

Develop domain theory, but constructively and predicatively in univalent foundations.

Why constructively and predicatively?

- More general than with classical reasoning
- Relevance in:
  - computer science (algorithm extraction);
  - pointfree/formal topology.
- No constructive justification of impredicativity axioms in univalent foundations (yet)

Aim and motivation

Develop domain theory, but constructively and predicatively in univalent foundations.

Why univalent foundations?

- Implemented in proof assistants
- Constructive and predicative by default
- Novel and natural interpretation of mathematical equality
  Avoids setoids

Aim

Develop domain theory, but constructively and predicatively in univalent foundations.

In the paper

- free dcpo with a least element on a set;
- exponentials
- (co)limits, including Scott's D_∞ model of the untyped λ-calculus;
- continuous and algebraic dcpos;
- rounded ideal completion.

Relation to other approaches

- Predicative domain theory has been studied previously by preferring information systems, abstract bases or formal topologies and approximable relations over dcpos and Scott continuous functions.

Relation to other approaches

- Predicative domain theory has been studied previously by preferring information systems, abstract bases or formal topologies and approximable relations over dcpos and Scott continuous functions.

Our approach studies dcpos directly and uses type universes to account for the fact that dcpos may be large.

Relation to other approaches

- Predicative domain theory has been studied previously by preferring information systems, abstract bases or formal topologies and approximable relations over dcpos and Scott continuous functions.

Our approach studies dcpos directly and uses type universes to account for the fact that dcpos may be large.

- Other constructive work (aimed the semantics of programming languages) is often impredicative and studies ω-chains.

Relation to other approaches

- Predicative domain theory has been studied previously by preferring information systems, abstract bases or formal topologies and approximable relations over dcpos and Scott continuous functions.

Our approach studies dcpos directly and uses type universes to account for the fact that dcpos may be large.

- Other constructive work (aimed the semantics of programming languages) is often impredicative and studies ω-chains.

We work predicatively and with the more general directed families rather than ω-chains, as we intend our theory to be also applicable to topology and algebra.

(Im)predicativity in UF

Definition (Subsingleton)

A type X is a *subsingleton* (or *proposition*, or *truth value*) if it has at most one element.

Definition ($\Omega_{\mathcal{U}}$)

Write $\Omega_{\mathcal{U}}$ for the type of *subsingletons in a universe* $\mathcal{U}$.
Note $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \text{is-subsingleton}(P)$ lives in $\mathcal{U}^+$.

(Im)predicativity in UF

Definition (Subsingleton)

A type X is a *subsingleton* (or *proposition*, or *truth value*) if it has at most one element.

Definition ($\Omega_{\mathcal{U}}$)

Write $\Omega_{\mathcal{U}}$ for the type of *subsingletons in a universe* $\mathcal{U}$.
Note $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \text{is-subsingleton}(P)$ lives in $\mathcal{U}^+$.

Definition (Has size)

Let $\mathcal{V}$ be a universe. A type X *has size* $\mathcal{V}$ if we have a type Y in $\mathcal{V}$ equivalent to X.

Definition (Impredicativity of $\mathcal{U}$)

We say that $\mathcal{U}$ is *impredicative* if $\Omega_{\mathcal{U}}$ has size $\mathcal{U}_0$.

Excluded middle implies impredicativity

Definition

Excluded middle (for a universe $\mathcal{U}$) says: every proposition $P : \mathcal{U}$ is either inhabited or empty.

Proposition

Excluded middle for $\mathcal{U}$ implies impredicativity of $\mathcal{U}$.

Proof.

With excluded middle for $\mathcal{U}$, there are only two propositions in $\mathcal{U}$, so that $\Omega_{\mathcal{U}} \simeq \mathbf{2} : \mathcal{U}_0$. □

Dcpos in type theory

- Main objects of domain theory: *directed complete posets (dcpos)*.
- In traditional set-theoretic foundations, a dcpo is a poset that has suprema of all directed subsets.

What is the right definition in predicative univalent foundations?

Dcpos in type theory

- Main objects of domain theory: *directed complete posets (dcpos)*.
- In traditional set-theoretic foundations, a dcpo is a poset that has suprema of all directed subsets.

What is the right definition in predicative univalent foundations?

Naive definition

A *poset* in $\mathcal{U}$ is a type $P : \mathcal{U}$ with a reflexive, transitive and antisymmetric relation $- \sqsubseteq - : P \to P \to \mathcal{U}$. Moreover, P should be a set and $p \sqsubseteq q$ should be a subsingleton for every $p, q : P$.

Dcpos in type theory

- Main objects of domain theory: *directed complete posets (dcpos)*.
- In traditional set-theoretic foundations, a dcpo is a poset that has suprema of all directed subsets.

What is the right definition in predicative univalent foundations?

Naive definition

A *poset* in $\mathcal{U}$ is a type $P : \mathcal{U}$ with a reflexive, transitive and antisymmetric relation $- \sqsubseteq - : P \to P \to \mathcal{U}$. Moreover, P should be a set and $p \sqsubseteq q$ should be a subsingleton for every $p, q : P$.
A *dcpo* is a poset that has suprema for every directed family $I \to P$ with indexing type $I : \mathcal{U}$.

The problem with the naive definition

In our constructive and predicative setting there are no nontrivial examples of the naive definition.

The problem with the naive definition

In our constructive and predicative setting there are no nontrivial examples of the naive definition.

- For instance, if $\mathbf{2}$ is directed complete, then weak excluded middle holds.
- The naive definition is not size-aware.

Small and large

For the poset $\Omega_{\mathcal{U}}$:

- the carrier lives in $\mathcal{U}^+$;
- the order has truth values in $\mathcal{U}$;
- suprema of directed families indexed by types in $\mathcal{U}$ exist.

Small and large

For the poset $\Omega_{\mathcal{U}}$:

- the carrier lives in $\mathcal{U}^+$;
- the order has truth values in $\mathcal{U}$;
- suprema of directed families indexed by types in $\mathcal{U}$ exist.

Categorically speaking: we have a large, but locally small, category with small filtered colimits (directed suprema).

Small and large

For the poset $\Omega_{\mathcal{U}}$:

- the carrier lives in $\mathcal{U}^+$;
- the order has truth values in $\mathcal{U}$;
- suprema of directed families indexed by types in $\mathcal{U}$ exist.

Categorically speaking: we have a large, but locally small, category with small filtered colimits (directed suprema).

This is typical for all our examples. But our definition is more general.

Size-aware definition

Definition (Official)

A *dcpo* is parameterized by three universes:

- the carrier lives in a universe $\mathcal{U}$;
- the order has truth values in a universe $\mathcal{T}$;
- suprema of directed families indexed by types in a universe $\mathcal{V}$ exist.

We will speak of $\mathcal{V}$-dcpos and leave $\mathcal{U}$ and $\mathcal{T}$ implicit.

Size-aware definition

Definition (Official)

A *dcpo* is parameterized by three universes:

- the carrier lives in a universe $\mathcal{U}$;
- the order has truth values in a universe $\mathcal{T}$;
- suprema of directed families indexed by types in a universe $\mathcal{V}$ exist.

We will speak of $\mathcal{V}$-dcpos and leave $\mathcal{U}$ and $\mathcal{T}$ implicit.

All our examples are locally small: the order has truth values of size $\mathcal{V}$ (but $\mathcal{T}$ is not necessarily the same as $\mathcal{V}$).

Examples

Example (The powerset as a dcpo)

Given a set $X : \mathcal{U}$, the *powerset* $\mathcal{P}(X) :\equiv X \to \Omega_{\mathcal{U}}$ is a $\mathcal{U}$-dcpo.
Its carrier lives in $\mathcal{U}^+$ and it is locally small.

Examples

Example (The powerset as a dcpo)

Given a set $X : \mathcal{U}$, the *powerset* $\mathcal{P}(X) :\equiv X \to \Omega_{\mathcal{U}}$ is a $\mathcal{U}$-dcpo.
Its carrier lives in $\mathcal{U}^+$ and it is locally small.

Example (The free dcpo with a least element on a set)

Given a set $X : \mathcal{U}$, we can construct the free $\mathcal{U}$-dcpo with a least element (as the *lifting* of X). Its carrier lives in $\mathcal{U}^+$ and it is locally small.

Exponentials and (co)limits

Taking exponentials of dcpos can increase universe levels in general.
Fortunately:

- If $D, E : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{U}}$ and $\mathcal{V} < \mathcal{U}$, then $E^D : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{U}}$.
  This allows us to construct the Scott model of PCF.

Exponentials and (co)limits

Taking exponentials of dcpos can increase universe levels in general.
Fortunately:

- If $D, E : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{U}}$ and $\mathcal{V} < \mathcal{U}$, then $E^D : \mathcal{V}\text{-DCPO}_{\mathcal{U},\mathcal{U}}$.
  This allows us to construct the Scott model of PCF.
- We can construct (co)limits predicatively, including Scott's D_∞, i.e. we have $D_\infty : \mathcal{U}_0\text{-DCPO}_{\mathcal{U}_1,\mathcal{U}_1}$ such that $D_\infty \cong D_\infty^{D_\infty}$.
  This gives the Scott model of the untyped λ-calculus.

Way-below relation

Definition (Way-below relation, $x \ll y$)

Let x and y be elements of a $\mathcal{V}$-dcpo D. Then x is *way below* y, written $x \ll y$, if for every directed family $\alpha : I \to D$ (with $I : \mathcal{V}$), the inequality $y \sqsubseteq \bigsqcup \alpha$ implies that there exists $i : I$ such that $x \sqsubseteq \alpha_i$ already.

Way-below relation

Definition (Way-below relation, $x \ll y$)

Let x and y be elements of a $\mathcal{V}$-dcpo D. Then x is *way below* y, written $x \ll y$, if for every directed family $\alpha : I \to D$ (with $I : \mathcal{V}$), the inequality $y \sqsubseteq \bigsqcup \alpha$ implies that there exists $i : I$ such that $x \sqsubseteq \alpha_i$ already.

Definition (Compact)

An element x of a $\mathcal{V}$-dcpo is *compact* if $x \ll x$.

Example (Compact elements in the powerset)

An element of the powerset $\mathcal{P}(X)$ of a set $X : \mathcal{U}$ is compact precisely when it is Kuratowski finite (i.e. finitely enumerable).

Bases and continuous dcpos

Classically, a continuous dcpo is a dcpo where every element is the directed join of the set of elements way below it.

Bases and continuous dcpos

Classically, a continuous dcpo is a dcpo where every element is the directed join of the set of elements way below it.

- Predicatively, if x is an element of a dcpo D, then $\sum_{y:D} y \ll x$ is typically large, so its directed join need not exist for size reasons.
- We also want to be able to give a satisfactory account of the rounded ideal completion.
- Our solution is to use a predicative version of bases.
- For the special case of algebraic dcpos, our situation is the poset analogue of accessible categories.

Bases and continuous dcpos

Classically, a continuous dcpo is a dcpo where every element is the directed join of the set of elements way below it.

Definition (Basis)

A *basis* for a $\mathcal{V}$-dcpo D is a function $\beta : B \to D$ with $B : \mathcal{V}$ such that for every $x : D$ there exists some $\alpha : I \to B$ with $I : \mathcal{V}$ such that

- $\beta \circ \alpha$ is directed and its supremum is x;
- $\beta(\alpha_i) \ll x$ for every $i : I$.

Bases and continuous dcpos

Classically, a continuous dcpo is a dcpo where every element is the directed join of the set of elements way below it.

Definition (Basis)

A *basis* for a $\mathcal{V}$-dcpo D is a function $\beta : B \to D$ with $B : \mathcal{V}$ such that for every $x : D$ there exists some $\alpha : I \to B$ with $I : \mathcal{V}$ such that

- $\beta \circ \alpha$ is directed and its supremum is x;
- $\beta(\alpha_i) \ll x$ for every $i : I$.

Moreover, we require that $\ll$ is small when restricted to the basis. That is, we have $\ll^B : B \to B \to \mathcal{V}$ such that $\Big(\beta(b) \ll \beta(b')\Big) \simeq \Big(b \ll^B b'\Big)$ for every $b, b' : B$.

Bases and continuous dcpos

Classically, a continuous dcpo is a dcpo where every element is the directed join of the set of elements way below it.

Definition (Basis)

A *basis* for a $\mathcal{V}$-dcpo D is a function $\beta : B \to D$ with $B : \mathcal{V}$ such that for every $x : D$ there exists some $\alpha : I \to B$ with $I : \mathcal{V}$ such that

- $\beta \circ \alpha$ is directed and its supremum is x;
- $\beta(\alpha_i) \ll x$ for every $i : I$.

Moreover, we require that $\ll$ is small when restricted to the basis. That is, we have $\ll^B : B \to B \to \mathcal{V}$ such that $\Big(\beta(b) \ll \beta(b')\Big) \simeq \Big(b \ll^B b'\Big)$ for every $b, b' : B$.

Definition (Continuous dcpo)

A dcpo is *continuous* if there exists some basis for it.

Algebraic dcpos and examples

Definition

A dcpo is *algebraic* if there exists some basis of compact elements for it.

Example (Powersets are algebraic)

The powerset $\mathcal{P}(X)$ of a set $X : \mathcal{U}$ is algebraic.
A compact basis is given by mapping lists on X to their corresponding Kuratowski finite subsets of X.

Example (The free dcpo with a least element is algebraic)

The free $\mathcal{U}$-dcpo with a least element on a set $X : \mathcal{U}$ is algebraic.
A compact basis is given by $\mathbf{1} + X$.

Example (D_∞ is algebraic)

Scott's D_∞ is algebraic.

Rounded ideal completion

Definition ($\mathcal{V}$-abstract basis)

A *$\mathcal{V}$-abstract basis* is a type $B : \mathcal{V}$ together with a truth valued relation $\prec : B \to B \to \mathcal{V}$ satisfying transitivity and (nullary & binary) interpolation.

Rounded ideal completion

Definition ($\mathcal{V}$-abstract basis)

A *$\mathcal{V}$-abstract basis* is a type $B : \mathcal{V}$ together with a truth valued relation $\prec : B \to B \to \mathcal{V}$ satisfying transitivity and (nullary & binary) interpolation.

Rounded ideal completion $\mathrm{Idl}(B, \prec)$

Starting with an $\mathcal{V}$-abstract basis $(B, \prec)$, the rounded ideal completion $\mathrm{Idl}(B, \prec)$ gives a locally small, continuous $\mathcal{V}$-dcpo with basis $\downarrow(-) : B \to \mathrm{Idl}(B, \prec)$.

Rounded ideal completion

Definition ($\mathcal{V}$-abstract basis)

A *$\mathcal{V}$-abstract basis* is a type $B : \mathcal{V}$ together with a truth valued relation $\prec : B \to B \to \mathcal{V}$ satisfying transitivity and (nullary & binary) interpolation.

Rounded ideal completion $\mathrm{Idl}(B, \prec)$

Starting with an $\mathcal{V}$-abstract basis $(B, \prec)$, the rounded ideal completion $\mathrm{Idl}(B, \prec)$ gives a locally small, continuous $\mathcal{V}$-dcpo with basis $\downarrow(-) : B \to \mathrm{Idl}(B, \prec)$.

Example (Rounded ideal completion of dyadics)

Inductively define a type $\mathbb{D} : \mathcal{U}_0$ and an order $\prec : \mathbb{D} \to \mathbb{D} \to \mathcal{U}_0$ representing dyadic rationals in the interval $(-1, 1)$.
Then $(\mathbb{D}, \prec)$ is a dense linear order without endpoints.
And $\mathrm{Idl}(\mathbb{D}, \prec) : \mathcal{U}_1$ is a continuous $\mathcal{U}_0$-dcpo with no compact elements (and so can't be algebraic).

Future and current work

Future and current work

- Predicative account of algebraic and continuous exponentials;
- Develop applications to topology and locale theory;
- Understand what classical theorems don't have constructive and predicative counterparts. E.g. Zorn's Lemma doesn't imply excluded middle, but it does imply impredicativity.

Conclusion

Conclusion

- We have developed domain theory constructively and predicatively in univalent foundations, including:
  - Scott's D_∞ model of the untyped λ-calculus;
  - continuous dcpos and the rounded ideal completion;
  - algebraic dcpos.
- We work predicatively by having large dcpos with joins of small directed families. Often these are locally small.
- Almost all our results are formalized in Agda.

Extended version of the CSL paper

TdJ and Martín Hötzel Escardó. *Domain Theory in Constructive and Predicative Univalent Foundations*. arXiv: 2008.01422 [math.LO].