Coherence for Segal types via the Leibniz adjunction

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Proof Systems in Actual Practice: Reasoning and Computation

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Introduction & first main result (informally)

- Riehl and Shulman introduced simplicial type theory (STT) as a synthetic framework for higher categories.[1]
- In homotopy type theory (HoTT): types behave like ∞-groupoids. Therefore, STT additionally equips types with a notion of directed morphisms, via an interval.

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- Segal types are those types in which such morphisms compose.
- Composition should be associative, identities should be neutral w.r.t. composition, and higher coherences should be satisfied, e.g. the different ways of rewriting $\mathrm{id} \circ (f \circ \mathrm{id})$ to f should be the same, up to a higher morphism.

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🏁 **For Segal types, all higher coherences can be derived.**

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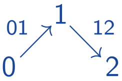
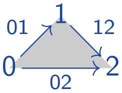
First main result (more precisely)

- Simplicial combinatorics: composition in X is given by extending along the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$.

$\Lambda^2_1 =$ 

$\Delta^2 =$ 

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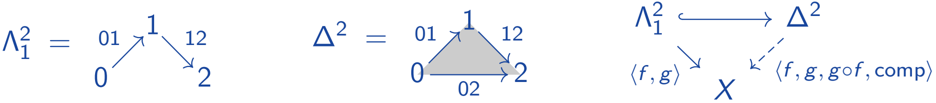

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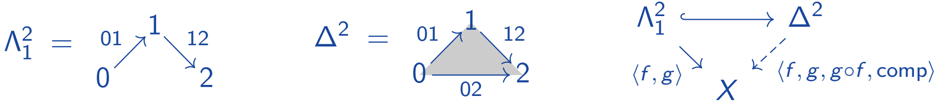

Def. A type X is a Segal type if every map $\Lambda^2_1 \to X$ has a unique (in the contractible/HoTT sense) extension to a map $\Delta^2 \to X$.

We also say that a Segal type has unique fillers for (2,1)-horns.

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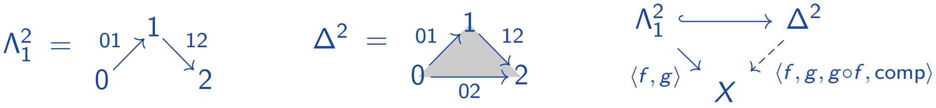

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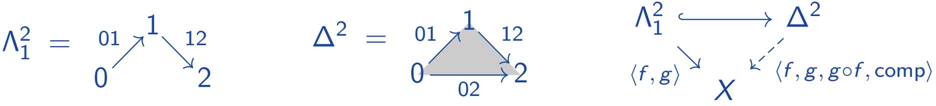

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Thm. Any Segal type has unique fillers for all (n,k)-horns where $0 < k < n$.

- This generalizes a result of Riehl & Shulman for $n = 3$ and $k \in \{1, 2\}$, and is a STT version of a result by Lurie.

Foundational setup

- Instead of simplicial type theory as introduced by Riehl & Shulman, we follow Gratzer, Weinberger and Buchholtz[2] and work in an axiomatic extension of HoTT:

$$\text{HoTT} + \text{bounded distributive lattice } (\mathbb{I}, 0, 1, \wedge, \vee).$$

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- David Wärn (Feb. 2026): any semi-lattice is automatically a set!

  https://martinescardo.github.io/TypeTopology/gist.ThereAreNoHigherSemilattices2.html

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Reduction to the Leibniz adjunction

- Segal types can be characterized by orthogonality

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What we did:

- Reduce simplicial comb. to formal arguments about $\pitchfork$ and $\widehat{\times}$ as much as possible.
- Prove the Leibniz adjunction for maps in HoTT.

Pushout-product and pullback-hom

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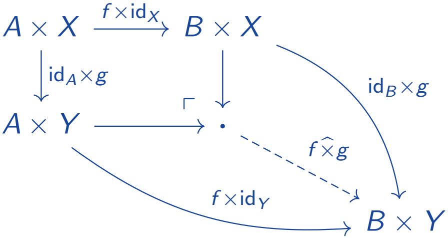

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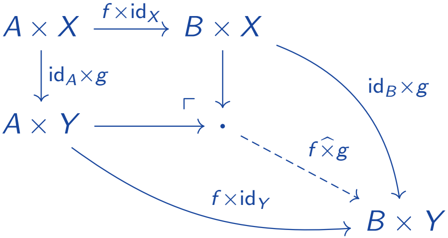

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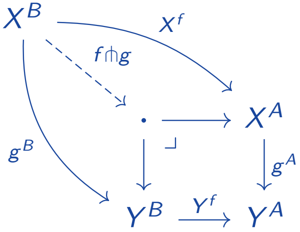

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Def. For $f : A \to B$ and $g : X \to Y$, write

$$\mathrm{Map}(f,g) :\equiv \left\{ \begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow{v} & Y \end{array} \right\}$$

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Thm. We have a natural equivalence

$$\mathrm{Map}(f \mathbin{\widehat{\times}} g, h) \simeq \mathrm{Map}(f, g \pitchfork h).$$

Towards a proof of the Leibniz adjunction

- Unfolding definitions, an element of $\mathrm{Map}(f \mathbin{\widehat{\times}} g, h)$ is a 7-tuple: three maps, three equations between maps and one additional coherence.

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$$\mathsf{Fam}((X, Y); (X', Y')) :\equiv \sum(m : X \to X') \prod(x : X) (Y\, x \to Y'(m\, x)).$$
- The above equivalence χ extends to an equivalence
$$\mathsf{Map}(f, g) \simeq \mathsf{Fam}(\chi\, f, \chi\, g).$$

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- Steps (1)–(3) turn out to be relatively easy and natural. ☺

Pushout-products and pullback-homs of families

- As observed in HoTT by Rijke,[3] the pushout-product $f \mathbin{\widehat{\times}} g$ of two maps is the fiberwise join, a certain pushout denoted by $*$.

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$$(A, B) \mathbin{\widehat{\times}} (X, Y) :\equiv (A \times X, (a, x) \mapsto B\, a * Y\, x).$$

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and
$$\mathsf{const}(f : C \to D) :\equiv \textstyle\sum(d : D) \prod(c : C)\, f\, c = d.$$

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- Recall that the pushout-product of families involved joins ($*$) and that pullback-homs involved a constancy condition (const).
- The Leibniz adjunction for families essentially follows from a small observation relating joins and constant maps.

Lemma For types X, Y and Z, we have equivalences

$$\begin{aligned}(X * Y \to Z) &\simeq \sum (f : X \to Z)\,(Y \to \mathsf{const}\, f) \\ &\simeq \sum (g : Y \to Z)\,(X \to \mathsf{const}\, g).\end{aligned}$$

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- The desire to formalize led to a much nicer proof.
- Families proved much more convenient than maps.
  Straightening vs unstraightening

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Conclusion

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The Leibniz adjunction in homotopy type theory, with an application to simplicial type theory. TdJ, Nicolai Kraus, Axel Ljungström. January 2026. `arXiv:2601.21843`.

Fully formalized in Cubical Agda.

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