

Epimorphisms and acyclic types

Tom de Jong¹

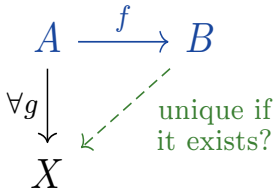
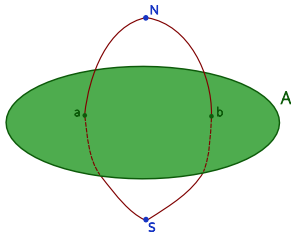
j.w.w. Ulrik Buchholtz¹ and Egbert Rijke²

¹University of Nottingham, UK

²University of Ljubljana, Slovenia

DutchCATS

20 March 2023



Homotopy type theory (HoTT)

- ▶ In HoTT, we think of **types as spaces**.
- ▶ If we have a type A with points $a, b : A$, then we may have identifications $p, q : a =_A b$ and **higher** identifications $\alpha, \beta : p =_{a=A} b q$, etc.
- ▶ A type is a **set** or **0-type** if there are no higher identifications. E.g. \mathbb{N} , $\mathbb{N} \rightarrow \mathbf{2}$, $\mathbb{N} \rightarrow \mathbb{N}$, etc. are all 0-types.

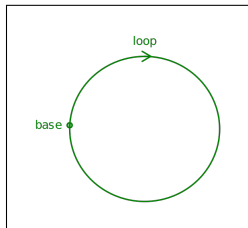
Higher types

- ▶ The circle \mathbb{S}^1

Higher inductive type

base : \mathbb{S}^1

loop : base = base



is a **1-type**: its identity types are 0-types.

Indeed, $\pi_1(\mathbb{S}^1, \text{base}) \simeq (\text{base} = \text{base}) \simeq \mathbb{Z}$.

- ▶ Similarly, we get the notion of a **k -type** for $k \geq 0$.
(Actually, $k \geq -2$.)

Epimorphisms

- ▶ In 1-category theory, a map $f : A \rightarrow B$ is an **epi(morphism)** if for every $g, h : B \rightarrow C$ we have

$$g \circ f = h \circ f \implies g = h.$$

In other words, $(-) \circ f$ is an **injection**.

- ▶ Note:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \forall g \downarrow & \swarrow & \\ X & & \end{array} \begin{array}{l} \text{unique if} \\ \text{it exists?} \end{array} \iff f \text{ is an epi}$$

- ▶ Def. A map $f : A \rightarrow B$ is an **epi** if $(-) \circ f$ is an **embedding**.

Epis w.r.t. k -types

- ▶ Def. (repeated) A map $f : A \rightarrow B$ is an **epi** if for every type X , the map

$$(B \rightarrow X) \xrightarrow{(-) \circ f} (A \rightarrow X)$$

is an embedding.

- ▶ Def. A map $f : A \rightarrow B$ is an **epi w.r.t. k -types** if for every k -type X , the map

$$(B \rightarrow X) \xrightarrow{(-) \circ f} (A \rightarrow X)$$

is an embedding.

- ▶ Lemma A map $f : A \rightarrow B$ is an epi w.r.t. k -types if and only if its k -truncation $\|f\|_k : \|A\|_k \rightarrow \|B\|_k$ is.

Basic properties of epis

- ▶ If $f : A \rightarrow B$ is an epi/ k -epi, then the composite $A \xrightarrow{f} B \xrightarrow{g} C$ is an epi/ k -epi if and only if g is.
- ▶ Every equivalence is an epi and every k -equivalence is a k -epi. Hence, every k -connected map is a k -epi.
- ▶ A map $f : A \rightarrow B$ is an epi if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & & \downarrow \text{id} \\ B & \xrightarrow{\text{id}} & B \end{array}$$

is a pushout.

Basic properties of epis

- ▶ If $f : A \rightarrow B$ is an epi/ k -epi, then the composite $A \xrightarrow{f} B \xrightarrow{g} C$ is an epi/ k -epi if and only if g is.
- ▶ Every equivalence is an epi and every k -equivalence is a k -epi. Hence, every k -connected map is a k -epi.
- ▶ A map $f : A \rightarrow B$ is an epi if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & & \downarrow \text{id} \\ B & \xrightarrow{\text{id}} & B \end{array}$$

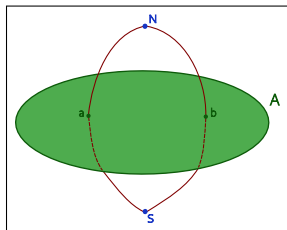
is a pushout.

- ▶ A map $f : A \rightarrow B$ is a 0-epi if and only if it is a **surjection**.
- ▶ To see what happens for $k > 0$, we turn to **acyclic types**.

Acyclic types

- Def. The **suspension** ΣA of a type A is the pushout

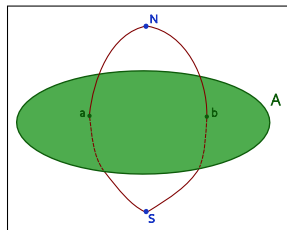
$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ \downarrow & \lrcorner & \downarrow N \\ \mathbf{1} & \xrightarrow{S} & \Sigma A \end{array}$$



Acyclic types

- ▶ Def. The **suspension** ΣA of a type A is the pushout

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ \downarrow & \lrcorner & \downarrow N \\ \mathbf{1} & \xrightarrow{S} & \Sigma A \end{array}$$



- ▶ Def. A type A is **acyclic** if ΣA is contractible, and **k -acyclic** if ΣA is k -connected (i.e. $\|\Sigma A\|_k$ is contractible).
- ▶ Ex. A type is 0-acyclic if and only if it is inhabited.
- ▶ Ex. Every k -connected type is $(k + 1)$ -acyclic, so the circle \mathbb{S}^1 is 1-acyclic.

Acyclic maps

- ▶ Def. A map $f : A \rightarrow B$ is (k -)acyclic if all of its fibres are. (Recall: $\text{fib}_f(b) \equiv \sum_{a:A} f(a) = b$.)
- ▶ Lemma A map $f : A \rightarrow B$ is acyclic/ k -acyclic if and only if its codiagonal ∇_f is an equivalence/ k -connected.

The diagram illustrates the relationship between a map $f : A \rightarrow B$ and its codiagonal ∇_f . It features three objects: A at the top left, B at the top right, and B at the bottom right. The object B at the bottom right is also the codomain of the map f from A . The map f is represented by a horizontal arrow from A to B . A vertical arrow labeled f points from A down to the B at the bottom left. A horizontal arrow labeled inl points from the B at the bottom left to the object $B \sqcup_A B$ in the middle. A vertical arrow labeled inr points from B at the top right down to $B \sqcup_A B$. A horizontal arrow labeled id points from B at the top right down to B at the bottom right. A dashed arrow labeled ∇_f points from $B \sqcup_A B$ to B at the bottom right. A curved arrow labeled id points from B at the top right down to B at the bottom right. A curved arrow labeled id points from the B at the bottom left to B at the bottom right.

Proof. For every $b : B$, we have $\sum \text{fib}_f(b) \simeq \text{fib}_{\nabla_f}(b)$. □

The epimorphisms are the acyclic maps

- Thm. A map is an (k -)epi if and only if it is (k -)acyclic.

Proof. $f : A \rightarrow B$ is an epi

$$\iff \begin{array}{ccc} A & \xrightarrow{f} & B \\ f \downarrow & & \downarrow \text{id} \\ B & \xrightarrow{\text{id}} & B \end{array} \text{ is a pushout}$$

$$\iff \nabla_f : B \sqcup_A B \rightarrow B \text{ is an equivalence}$$

$$\iff f \text{ is acyclic.} \quad \square$$

Perfect groups and k -acyclic sets

- ▶ Def. A (set-based) group G is **perfect** if its **abelianisation** is trivial. E.g., the group A_5 of even permutations on a 5-element set is perfect.

- ▶ Thm. For a group G , its **classifying type** BG is 2-acyclic if and only if G is perfect.

Perfect groups and k -acyclic sets

- ▶ Def. A (set-based) group G is **perfect** if its **abelianisation** is trivial. E.g., the group A_5 of even permutations on a 5-element set is perfect.
 - ▶ Thm. For a group G , its **classifying type** BG is 2-acyclic if and only if G is perfect.
-

- ▶ Prop. A **set** is 1-acyclic if and only if it is contractible.

Proof. Let G be the **free group** on a 1-acyclic set A with inclusion of generators $\eta : A \hookrightarrow G$. If A is 1-acyclic, then $A \rightarrow \mathbf{1}$ is a 1-epi, so the constant map

$$BG \rightarrow (A \rightarrow BG)$$

is an embedding. Hence, the constant map $G \rightarrow (A \rightarrow G)$ is an equivalence. Thus, η is constant. But it is also an embedding, so A must be a subsingleton. Finally, A is also inhabited, because it is 0-acyclic. □

Characterising 1-acyclic and 2-acyclic types

- ▶ Thm. A type is 1-acyclic if and only if it is connected.

Proof. We already know that k -connected types are $(k + 1)$ -acyclic, so every connected type is 1-acyclic. Conversely, if A is 1-acyclic, then the composite

$$A \xrightarrow{|-\|_0} \|A\|_0 \rightarrow \mathbf{1}$$

is a 1-epi. Further, $|-\|_0$ is connected and hence a 1-epi. Thus, $\|A\|_0 \rightarrow \mathbf{1}$ is a 1-epi and $\|A\|_0$ is 1-acyclic. But this means that the set $\|A\|_0$ is contractible by the previous proposition. Hence, A is connected. □

- ▶ Cor. Every k -acyclic type is connected for $k \geq 1$.
- ▶ Thm. A type A is 2-acyclic if and only if connected and $\pi_1(A, a)$ is perfect for every $a : A$.

Acyclic types and the Freudenthal suspension theorem

- ▶ Thm. Every 1-connected acyclic type is ∞ -connected.

Proof. By the **Freudenthal suspension theorem**, the unit $\sigma : A \rightarrow \Omega\Sigma A$ of the loop-suspension adjunction is $2n$ -connected whenever A is n -connected (for $n \geq 0$).

If A is acyclic, then $\Sigma A \simeq \mathbf{1}$, so $\Omega\Sigma A \simeq \mathbf{1}$, so the connectedness of σ is that of A .

Now if A is also 1-connected, then σ , and hence A , is in turn 2-connected, then 4-connected, etc., hence 2^n -connected for all n . □

- ▶ Thm. A 1-connected type is $(k + 1)$ -acyclic if and only if it is k -connected.

The Higman group: a nontrivial acyclic type

- ▶ The **Higman group** is defined as the group with 4 generators a, b, c, d and 4 relations

$$r_a : a = [d, a] \quad r_b : b = [a, b] \quad r_c : c = [b, c] \quad r_d : d = [c, d],$$

where $[x, y] \equiv xyx^{-1}y^{-1}$ denotes the **commutator**.

- ▶ In HoTT we can describe its **classifying type** BH as a HIT:

$$\text{pt} : BH$$

$$a, b, c, d : \text{pt} = \text{pt}$$

$$r_a : a = [d, a]$$

$$r_b : a = [a, b]$$

$$r_c : a = [b, c]$$

$$r_d : a = [c, d]$$

Contractibility of ΣBH (1)

- ▶ We describe ΣBH as a HIT and simplify its description step-by-step.

$$\begin{array}{l} N, S : \Sigma BH \\ m_{\text{pt}} : N = S \\ \vdots \end{array}$$

Contractibility of ΣBH (1)

- ▶ We describe ΣBH as a HIT and simplify its description step-by-step.

$$\begin{array}{l} N, S : \Sigma BH \\ m_{\text{pt}} : N = S \\ \vdots \end{array}$$

Contr. at (N, m_{pt})
~~~~~  
→

$$\begin{array}{l} N : \Sigma BH \\ m_a, m_b, m_c, m_d : \text{refl}_N = \text{refl}_N \\ m_{r_a} : m_a = [m_d, m_a] \\ m_{r_b} : m_b = [m_a, m_b] \\ m_{r_c} : m_c = [m_b, m_c] \\ m_{r_d} : m_d = [m_c, m_d] \end{array}$$

## Contractibility of $\Sigma BH$ (2)

$$N : \Sigma BH$$

$$m_a : \text{refl}_N = \text{refl}_N$$

$$m_b : \text{refl}_N = \text{refl}_N$$

$$\vdots$$

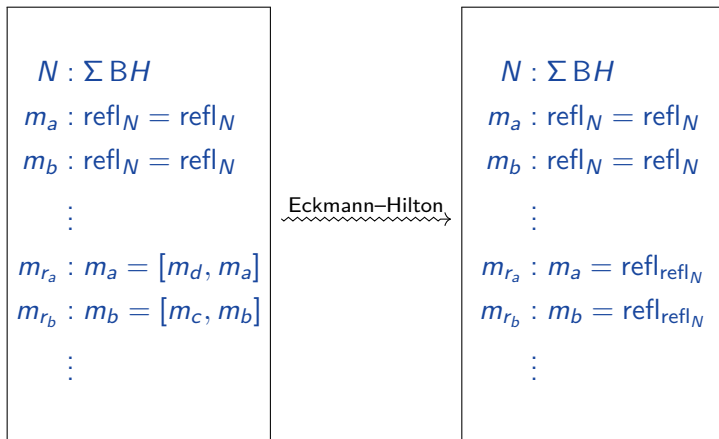
$$m_{r_a} : m_a = [m_d, m_a]$$

$$m_{r_b} : m_b = [m_c, m_b]$$

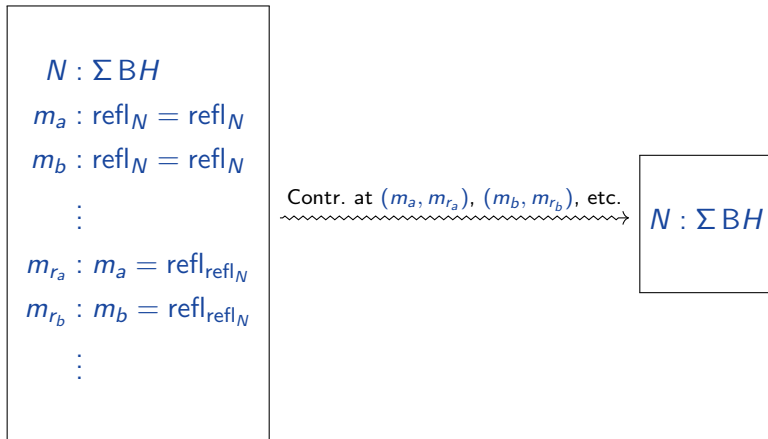
$$\vdots$$

## Contractibility of $\Sigma BH$ (2)

- ▶ The crux is that **commutators** are trivial in higher homotopy groups by the **Eckmann-Hilton** argument.



## Contractibility of $\Sigma BH$ (3)



- ▶ So  $\Sigma BH$  is equivalent to a single point and hence contractible, i.e.  $BH$  is acyclic.

# Nontriviality of the Higman group

- ▶ We can repeat the above argument for  $n$  generators and  $n$  relations, yielding an acyclic type for all  $n$ .
- ▶ For  $n \leq 3$ , the resulting groups turn out to be trivial.
- ▶ The **Higman group** ( $n = 4$ ) is in fact infinite but proving this seems to require a bit of group theory.

# Summary

At higher types the notion of **epimorphism**

- ▶ becomes quite strong,
- ▶ coincides with the notion of an **acyclic** map, and
- ▶ is interesting from the p.o.v. of **synthetic homotopy theory**.

## Future work

- ▶ Prove that  $BH$  is a 1-type and nontrivial with a proof on the level of classifying types.
- ▶ Do the acyclic maps form an **accessible modality**?  
(Classically, they do.)
- ▶ **Plus construction** in HoTT
- ▶ **Kan-Thurston** theorem in HoTT: every  $\infty$ -group can be presented by a pair  $(G, P)$  of a group  $G$  and perfect normal subgroup  $P \triangleleft G$
- ▶ In further developments, can we work around needing **Whitehead's principle** (every  $\infty$ -connected type is contractible) or the weaker principle that every 1-connected acyclic type is contractible?
- ▶ Use the theory of **binate groups** to prove acyclicity of some infinitely presented groups
- ▶ Applications (where surjectivity is not sufficient)



# References

- [1] G. Baumslag, E. Dyer and A. Heller. “The topology of discrete groups”. In: *J. Pure Appl. Algebra* 16.1 (1980), pp. 1–47. DOI: 10.1016/0022-4049(80)90040-7.
- [2] A. J. Berrick. “Universal groups, binate groups and acyclicity”. In: *Group Theory. Proceedings of the Singapore Group Theory Conference held at the National University of Singapore, June 8–19, 1987*. Ed. by Kai N. Cheng and Yu K. Leong. De Gruyter Proceedings in Mathematics. 1989. DOI: 10.1515/9783110848397-019.
- [3] A. J. Berrick and Carles Casacuberta. “A universal space for plus-constructions”. In: *Topology* 38.3 (1999), pp. 467–477. DOI: 10.1016/S0040-9383(97)00073-6.
- [4] Marc Bezem, Thierry Coquand, Peter Dybjer and Martín Escardó. “Free groups in HoTT/UF in Agda”. <https://www.cs.bham.ac.uk/~mhe/TypeTopology/FreeGroup.html>. 2021.
- [5] J. Daniel Christensen and Egbert Rijke. “Characterizations of modalities and lex modalities”. In: *Journal of Pure and Applied Algebra* 226.3, 106848 (2022). DOI: 10.1016/j.jpaa.2021.106848.
- [6] P. Hall. “Some constructions for locally finite groups”. In: *J. London Math. Soc.* 34 (1959), pp. 305–319. DOI: 10.1112/jlms/s1-34.3.305.
- [7] Pierre de la Harpe and Dusa McDuff. “Acyclic groups of automorphisms”. In: *Comment. Math. Helv.* 58.1 (1983), pp. 48–71. DOI: 10.1007/BF02564624.
- [8] Jean-Claude Hausmann and Dale Husemoller. “Acyclic maps”. In: *L’enseignement Mathématique* 25.1–2 (1979), pp. 53–75. DOI: 10.5169/seals-50372.
- [9] Graham Higman. “A finitely generated infinite simple group”. In: *J. London Math. Soc.* 26 (1951), pp. 61–64. DOI: 10.1112/jlms/s1-26.1.61.
- [10] Marc Hoyois. “On Quillen’s plus construction”. 2019. URL: <https://hoyois.app.uni-regensburg.de/papers/acyclic.pdf>.
- [11] D. M. Kan and W. P. Thurston. “Every connected space has the homology of a  $K(\pi, 1)$ ”. In: *Topology* 15.3 (1976), pp. 253–258. DOI: 10.1016/0040-9383(76)90040-9.
- [12] Ray Mines, Fred Richman and Wim Ruitenburg. *A Course in Constructive Algebra*. Universitext. Springer, 1988. DOI: 10.1007/978-1-4419-8640-5.

# References II

- [13] Brian A. Munson and Ismar Volić. *Cubical Homotopy Theory*. Vol. 25. New Mathematical Monographs. Cambridge University Press, 2015. DOI: [10.1017/CB09781139343329](https://doi.org/10.1017/CB09781139343329).
- [14] George Raptis. "Some characterizations of acyclic maps". In: *J. Homotopy Relat. Struct.* 14.3 (2019), pp. 773–785. DOI: [10.1007/s40062-019-00231-6](https://doi.org/10.1007/s40062-019-00231-6).
- [15] Egbert Rijke. "Introduction to Homotopy Type Theory". Book draft; version of 8 April. 2022. URL: <https://raw.githubusercontent.com/martinescardo/HoTTEST-Summer-School/main/HoTT/hott-intro.pdf>.
- [16] Luis Scoccola. "Nilpotent types and fracture squares in homotopy type theory". In: *Math. Structures Comput. Sci.* 30.5 (2020), pp. 511–544. DOI: [10.1017/s0960129520000146](https://doi.org/10.1017/s0960129520000146).
- [17] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: <https://homotopytypetheory.org/book>, 2013.