



**(Counter)examples of injective types**

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Dutch Categories And Types Seminar  
DutchCATS — Delft, 10 April 2026

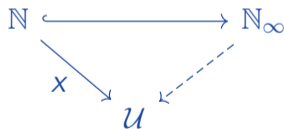
## Motivation

- ▶ We work in **univalent foundations** a.k.a. **homotopy type theory (HoTT)**.
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**Use of injectivity:** Given a sequence  $(X_n)_{n:\mathbb{N}}$  of searchable types, we use the injectivity of the universe to construct a new (searchable) type  $X_\infty$  by extending along  $\mathbb{N} \hookrightarrow \mathbb{N}_\infty$  and evaluating at  $\infty : \mathbb{N}_\infty$ .

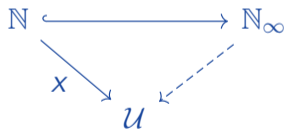


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- ▶ In this talk, we present new examples and counterexamples of injective types.

## Injective types

- Def. A type  $D$  is (algebraically) **injective** if for every *embedding*  $j : X \hookrightarrow Y$ , any map  $f : X \rightarrow D$  into  $D$  has a designated extension  $f/j$ .

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ & \searrow f & \swarrow f/j \\ & D & \end{array} \qquad (f/j \circ j = f)$$

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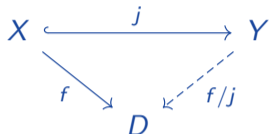
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- ▶ Recall: **embedding**  $\approx$  homotopically well-behaved injection.  
More precisely,  $j$  is an embedding if the canonical map  $x = x' \rightarrow jx = jx'$  is an equivalence, or equivalently, if the fibers of  $j$  are propositions.

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More precisely,  $j$  is an embedding if the canonical map  $x = x' \rightarrow jx = jx'$  is an equivalence, or equivalently, if the fibers of  $j$  are propositions.
- ▶ The notion of injectivity is sensitive to universe levels, so we really study  $\mathcal{U}, \mathcal{V}$ -injective types where  $X : \mathcal{U}$  and  $Y : \mathcal{V}$ , but we largely ignore this in this talk.

## Injective types can extend partial elements

- ▶ Let  $D$  be an injective type and  $P$  an arbitrary proposition. Suppose we are given a **partial element**  $f : P \rightarrow D$  of  $D$ .  
By injectivity, we can extend it to a total element  $d : D$ , as in

$$\begin{array}{ccc} P & \hookrightarrow & \mathbf{1} \\ & \searrow f & \swarrow d \\ & & D \end{array}$$

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- ▶ Types that can extend partial elements to total elements are called **flabby**.
- ▶ Clearly, injective types are flabby. But the converse holds too!

In short, extend  $f : X \rightarrow D$  along an embedding  $j : X \hookrightarrow Y$  by considering the propositional(!) fibers  $j^{-1} y := \sum x : X, j x = y$  and  $f \circ \text{pr}_1 : j^{-1} y \rightarrow D$ .

## Examples of injective types

- ▶ Any univalent universe  $\mathcal{U}$
- ▶ The type  $\Omega_{\mathcal{U}}$  of propositions in a universe  $\mathcal{U}$
- ▶ The type  $\mathcal{L}X := \Sigma(P : \Omega_{\mathcal{U}}), (P \rightarrow X)$  of partial elements of a type  $X : \mathcal{U}$
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### New examples

- ▶ The type of iterative (multi)sets in  $\mathcal{U}$
- ▶ The types of small  $\infty$ -magmas, monoids and groups
- ▶ The underlying set of any sup-complete poset, or more generally, of any pointed dcpo

## Injective dependent sums

- ▶ The injectivity of  $\infty$ -magmas, monoids and groups, follows from the injectivity of univalent universes via **sufficient criterion for the injectivity of  $\Sigma$ -types**.

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- ▶ For **subtypes** there is a necessary and sufficient criterion:

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- ▶ Ex. The injectivity of  $\Omega_{\mathcal{U}}$  follows by taking  $P := \text{is-prop}$  and the retraction given by the propositional truncation.

This generalizes to any **reflective subuniverse**.

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This is no coincidence:

- ▶ Thm. If there is a  $\mathcal{U}, \mathcal{U}$ -injective type in  $\mathcal{U}$  with two distinct points, then the type  $\Omega_{\neg\neg} := \Sigma(P : \Omega_{\mathcal{U}}) \times (\neg\neg P \rightarrow P)$  of  $\neg\neg$ -stable propositions in  $\mathcal{U}$ , whose native universe is  $\mathcal{U}^+$ , is equivalent to a type in  $\mathcal{U}$ .

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- ▶ The conclusion of the theorem, the resizing of  $\Omega_{\neg\neg}$ , is **not provable** in univalent foundations. This follows from a proof-theoretic argument due to Andrew Swan.
- ▶ This theorem is comparable to a result of Aczel, van den Berg, Granström & Schuster: in the predicative set theory **CZF** it is consistent that the only injective *sets* (as opposed to *classes*) are singletons.

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- ▶ But there are plenty of examples of types that cannot be shown to be injective in constructive mathematics, because their injectivity implies a **constructive taboo**: a statement that is not constructively provable and is false in some models.
- ▶ The relevant taboo in this case is **weak excluded middle**: for any proposition  $P$ , either  $\neg P$  or  $\neg\neg P$  holds. This is equivalent to De Morgan's law.

## Counterexamples of injective types I

If any of the following types is injective, then weak excluded middle holds.

- ▶ The type of booleans  $\mathbf{2} := \mathbf{1} + \mathbf{1}$ .
- ▶ The simple types, obtained from  $\mathbb{N}$  by iterating function types.
- ▶ The type of Dedekind reals.
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- ▶ The type of conatural numbers  $\mathbb{N}_\infty$ .
- ▶ More generally, any type with an **apartness relation** and two points apart.

Recall: apartness relation  $\approx$  positive (constructive) strengthening of  $\neq$ .

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- ▶ Rice-like theorem: Injective types have **no non-trivial decidable properties**.

Thm. If an injective type has a **decomposition**, then weak excluded middle holds.

A *decomposition* of a type  $X$  is defined to be a function  $f : X \rightarrow \mathbf{2}$  such that we have  $x_0 : X$  and  $x_1 : X$  with  $f x_0 = 0$  and  $f x_1 = 1$ .

## Counterexamples of injective types II

- ▶ While the type  $\Sigma(X : \mathcal{U}), X$  of **pointed** types and the type  $\Sigma(X : \mathcal{U}), \neg\neg X$  of **non-empty** types are both injective, the type of **inhabited** types need not be. Prop. The type  $\Sigma(X : \mathcal{U}), \|X\|$  of inhabited types is injective if and only if **all propositions are projective** (a weak choice principle).

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Prop. If the type  $\mathbb{R}P^\infty := \Sigma(X : \mathcal{U}_0), \|X \simeq \mathbf{2}\|$  of two-element types (a.k.a. infinite real projective space) is injective, then Fourman–Ščedrov's *world's simplest axiom of choice (WSAC)* holds.

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WSAC: “choosing an element of a two-element set at most one (subsingleton-many) time(s)”.

- ▶ Note that we have:

excluded middle  $\implies$  all propositions are projective  $\implies$  WSAC.

## Conclusion & future work

- ▶ There are plenty of (counter)examples of injective types in univalent foundations!
- ▶ Our results on injectives generalize specific observations, e.g. the universe cannot have a nontrivial apartness, ordinals have no non-trivial decidable properties.

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 *Examples and counterexamples of injective types*. TdJ, Martín Hötzel Escardó  
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Thank you!

# A peak at the Agda formalization

```
module Examples-4-13-a where
open import InjectiveTypes.MathematicalStructures ua
```

```
[1] : ainjective-type (Pointed-type  $\mathcal{U}$ )  $\mathcal{U}$   $\mathcal{U}$ 
[1] = ainjectivity-of-type-of-pointed-types
```

```
[2] : ainjective-type ( $\infty$ -Magma  $\mathcal{U}$ )  $\mathcal{U}$   $\mathcal{U}$ 
[2] = ainjectivity-of- $\infty$ -Magma
```

```
[3] : ainjective-type ( $\infty$ -Magma•  $\mathcal{U}$ )  $\mathcal{U}$   $\mathcal{U}$ 
[3] = ainjectivity-of- $\infty$ -Magma•
```

```
[4] : ainjective-type (monoid.Monoid { $\mathcal{U}$ })  $\mathcal{U}$   $\mathcal{U}$ 
[4] = ainjectivity-of-Monoid
```

```
[5] : ainjective-type (group.Group { $\mathcal{U}$ })  $\mathcal{U}$   $\mathcal{U}$ 
[5] = ainjectivity-of-Group
```

```
Counterexample-7-1 : ainjective-type 2  $\mathcal{U}$   $\mathcal{U}$   $\leftrightarrow$  typal-WEM  $\mathcal{U}$ 
Counterexample-7-1 = 2-ainjective-gives-WEM , WEM-gives-2-ainjective
```

```
Lemma-7-2 : WLPO  $\leftrightarrow$  ( $\Sigma$  f : ( $\mathbb{N}^\infty \rightarrow \mathbb{N}^\infty$ ) , ((n :  $\mathbb{N}$ )  $\rightarrow$  f (1 n) = 1 0)  $\times$  (f  $\infty$  = 1 1))
Lemma-7-2 = WLPO-is-discontinuous' ,
            ( $\lambda$  (f , p)  $\rightarrow$  basic-discontinuity-taboo' f p)
```

```
Counterexample-7-3-1 : ainjective-type  $\mathbb{N}^\infty$   $\mathcal{U}_0$   $\mathcal{U}_0 \rightarrow$  WLPO
Counterexample-7-3-1 =  $\mathbb{N}^\infty$ -injective-gives-WLPO
```

```
Counterexample-7-3-2 : ainjective-type  $\mathbb{N}^\infty$   $\mathcal{U}$   $\mathcal{V} \rightarrow$  typal-WEM  $\mathcal{U}$ 
Counterexample-7-3-2 =  $\mathbb{N}^\infty$ -injective-gives-WEM
```

```
Counterexample-7-4-1 : ainjective-type  $\mathbb{R}$   $\mathcal{U}_1$   $\mathcal{U}_1$ 
                     $\rightarrow \Sigma$  H : ( $\mathbb{R} \rightarrow \mathbb{R}$ ) ,
                    ((x :  $\mathbb{R}$ )  $\rightarrow$  (x < 0 $\mathbb{R}$   $\rightarrow$  H x = 0 $\mathbb{R}$ )
                     $\times$  (x  $\geq$  0 $\mathbb{R}$   $\rightarrow$  H x = 1 $\mathbb{R}$ ))
```

```
Counterexample-7-4-1 =  $\mathbb{R}$ -ainjective-gives-Heaviside-function
```

```
Counterexample-7-4-2 : ainjective-type  $\mathbb{R}$   $\mathcal{U}$   $\mathcal{V} \rightarrow$  typal-WEM  $\mathcal{U}$ 
Counterexample-7-4-2 =  $\mathbb{R}$ -ainjective-gives-WEM
```

# References

-  Aczel, Peter, Benno van den Berg, Johan Granström and Peter Schuster (2012). 'Are There Enough Injective Sets?' In: *Studia Logica* 101.3, pp. 467–482. DOI: 10.1007/s11225-011-9365-8.
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