Predicative Aspects of Order Theory in Univalent Foundations

Tom de Jong[†] Martín Hötzel Escardó[†]

[†]University of Birmingham, United Kingdom

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Introduction

- In the Scott model of PCF in HoTT/UF, the directed complete posets (dcpos) interpreting PCF types are large.
  E.g. $\mathbb{N}$ is in $\mathcal{U}_0$, but $[\![\mathtt{nat}]\!]$ is in $\mathcal{U}_1$.
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- In pointfree topology in HoTT/UF: the locales/frames are large.

We show that this largeness is unavoidable in predicative HoTT/UF.

Key result in this talk

Theorem (crude formulation)

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Theorem (Freyd) for comparison

A category with small (co)limits is small if and only if it is a poset.

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Various kinds of posets can only be small in impredicative HoTT/UF, *unless they are trivial.*

Ingredients to be made precise

- HoTT/UF
- Impredicativity
- Various kinds of posets
- Trivial
- Small poset

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The precise formulation will be a theorem of HoTT/UF. We do not make reference to models.

Univalent Foundations (UF) / Homotopy Type Theory (HoTT)

- UF: Intensional Martin-Löf Type Theory with Σ, Π, $+$, $\mathbf{0}$, $\mathbf{1}$, $\mathbb{N}$, Id, universes and propositional truncations ($\exists$).
- Some of our results use univalence.
- We assume function extensionality and propositional extensionality, which are implied by univalence.

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- Some of our results use univalence.
- We assume function extensionality and propositional extensionality, which are implied by univalence.
- Voevodsky assumed resizing axioms, but it is not known whether computational interpretations of these axioms are possible.

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Impredicativity in UF

Definition

A *proposition* (*subsingleton*, *truth value*) is a type with at most one element.

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The type $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \text{is-prop}(P)$ is the type of all propositions in a universe $\mathcal{U}$.

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It's important to notice that $\Omega_{\mathcal{U}}$ lives in the next universe $\mathcal{U}^+$.

Definition

A type $X : \mathcal{U}^+$ is *small* if we have $Y : \mathcal{U}$ with $Y \simeq X$.

Definition

The axiom $\Omega_{\mathcal{U}}\text{-Resizing}$ asserts that $\Omega_{\mathcal{U}}$ is small.

Excluded middle implies impredicativity

Definition

Excluded middle holds in a universe $\mathcal{U}$ if every proposition in $\mathcal{U}$ is either inhabited or empty.

Proposition

Excluded middle in $\mathcal{U}$ implies $\Omega_{\mathcal{U}}$-Resizing.

Proof.

With excluded middle in $\mathcal{U}$ we have $\Omega_{\mathcal{U}} \simeq \mathbf{1} + \mathbf{1}$. □

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So in studying impredicativity we *must* work constructively, i.e. without excluded middle.

Weak impredicativity in UF

Definition

- A proposition P is *$\neg\neg$-stable* if $\neg\neg P$ implies P.
- The type $\Omega_{\mathcal{U}}^{\neg\neg}$ is the type of all $\neg\neg$-stable propositions in a universe $\mathcal{U}$.
- The axiom $\Omega_{\mathcal{U}}^{\neg\neg}\text{-Resizing}$ asserts that $\Omega_{\mathcal{U}}^{\neg\neg}$ is small.

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Definition

Weak excluded middle holds in a universe $\mathcal{U}$ if for every proposition P in $\mathcal{U}$ either $\neg\neg P$ holds or $\neg P$ does.

Proposition

Weak excluded middle in $\mathcal{U}$ implies $\Omega_{\mathcal{U}}^{\neg\neg}\text{-Resizing}$.

Making the theorem precise

Theorem (crude formulation)

Various kinds of posets can only be small in impredicative HoTT/UF, *unless they are trivial.*

Ingredients to be made precise

- ✓ HoTT/UF
- ✓ Impredicativity: $\Omega_{\mathcal{U}}$-Resizing and $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing
  $\Omega_{\mathcal{U}}$ and $\Omega_{\mathcal{U}}^{\neg\neg}$ are small
- Various kinds of posets
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Posets in UF

Definition

A *poset* is type X with a binary proposition-valued relation $\sqsubseteq : X \to X \to \Omega_{\mathcal{T}}$ that is reflexive, antisymmetric and transitive.
(It follows that X is a *set* in the sense of UF.)

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A poset $(X, \sqsubseteq)$ is a *$\mathcal{U}$-sup-lattice* if every family $I \to X$ with $I : \mathcal{U}$ has a supremum in X.
The carrier X and the values of $\sqsubseteq$ are **not** required to be in $\mathcal{U}$ or even in the same universe.

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Definition

A poset X is a *$\mathcal{U}$-dcpo* if every *directed* family $I \to X$ with $I : \mathcal{U}$ has a supremum in X.

Examples of $\mathcal{U}$-sup-lattices

Example

The powerset $\mathcal{P}(X) :\equiv X \to \Omega_{\mathcal{U}}$ of $X : \mathcal{U}$ is $\mathcal{U}$-sup-lattice.

- If $A, B \in \mathcal{P}(X)$, then $A \sqsubseteq B \iff \forall_{x \in X} A(x) \to B(x)$.
- If $I : \mathcal{U}$ and $A_{(-)} : I \to \mathcal{P}(X)$, then its supremum is the subset $x \mapsto \exists_{i:I} A_i(x)$.
- Note: $\exists_{i:I} A_i(x) : \mathcal{U}$, but $\mathcal{P}(X) : \mathcal{U}^+$.

Example

The type $\Omega_{\mathcal{U}}$ is a $\mathcal{U}$-sup-lattice ordered by implication and with suprema given by existential quantification.

Examples of $\mathcal{U}$-dcpos

Example

For any set $X : \mathcal{U}$, the *lifting* $\mathcal{L}(X) :\equiv \sum_{P:\Omega_{\mathcal{U}}}(P \to X)$ of X is a $\mathcal{U}$-dcpo which lives in $\mathcal{U}^+$.

- Any element $x : X$ gives an element in $\mathcal{L}(X)$ by taking the proposition P to be $\mathbf{1}$.
- In general, P is the domain of definition of the partial element.

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In particular, for the Scott model of PCF:

- $[\![\mathtt{nat}]\!] \equiv \mathcal{L}(\mathbb{N})$ is a $\mathcal{U}_0$-dcpo in $\mathcal{U}_1$.
- $[\![\mathtt{nat} \Rightarrow \mathtt{nat}]\!]$ is the $\mathcal{U}_0$-dcpo of Scott continuous functions from $\mathcal{L}(\mathbb{N})$ to $\mathcal{L}(\mathbb{N})$, which lives in $\mathcal{U}_1$ again.

$\delta_{\mathcal{U}}$-completeness

Definition

A poset $(X, \sqsubseteq)$ is *$\delta_{\mathcal{U}}$-complete* if for every proposition $P : \mathcal{U}$ and elements $x \sqsubseteq y$, the family

$$\begin{aligned} \delta_{x,y,P} : \mathbf{1} + P &\to X \\ \mathrm{inl}(\star) &\mapsto x; \\ \mathrm{inr}(p) &\mapsto y; \end{aligned}$$

has a supremum $\bigvee \delta_{x,y,P}$ in X.

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- With excluded middle in $\mathcal{U}$, every poset is $\delta_{\mathcal{U}}$-complete.
- Assuming $x \neq y$, we have $\bigvee \delta_{x,y,P} = x \iff \neg P$, but $P \Rightarrow \bigvee \delta_{x,y,P} = y \Rightarrow \neg\neg P$.

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- With excluded middle in $\mathcal{U}$, every poset is $\delta_{\mathcal{U}}$-complete.
- Assuming $x \neq y$, we have $\bigvee \delta_{x,y,P} = x \iff \neg P$, but $P \Rightarrow \bigvee \delta_{x,y,P} = y \Rightarrow \neg\neg P$.
- If the two-element poset with $0 \sqsubseteq 1$ is $\delta_{\mathcal{U}}$-complete, then weak excluded middle holds in $\mathcal{U}$.

Examples of $\delta_{\mathcal{U}}$-complete posets

$\mathcal{U}$-sup-lattices (posets with all $\mathcal{U}$-suprema) are $\delta_{\mathcal{U}}$-complete, and so are $\mathcal{U}$-dcpos and $\mathcal{U}$-bounded complete posets.
(The family $\delta_{x,y,P}$ is bounded and directed when $x \sqsubseteq y$.)

Example

The $\mathcal{U}$-sup-lattices $\Omega_{\mathcal{U}}$ and $\mathcal{P}(X)$ for $X : \mathcal{U}$ are $\delta_{\mathcal{U}}$-complete.

Example

The $\mathcal{U}_0$-dcpos in the Scott model of PCF are $\delta_{\mathcal{U}_0}$-complete.

Making the theorem precise

Theorem (crude formulation)

Various kinds of posets can only be small in impredicative HoTT/UF, *unless they are trivial.*

Ingredients to be made precise

- ✓ HoTT/UF
- ✓ Impredicativity: $\Omega_{\mathcal{U}}$-Resizing and $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing
  $\Omega_{\mathcal{U}}$ and $\Omega_{\mathcal{U}}^{\neg\neg}$ are small
- ✓ Various kinds of posets: $\delta_{\mathcal{U}}$-complete posets
- Trivial
- Small poset

Nontriviality and positivity

Definition

A poset is *nontrivial* if we have $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$.

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- For $\delta_{\mathcal{U}}$-complete posets we can do better.

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Definition

An element x of a $\delta_{\mathcal{U}}$-complete poset is *strictly below* an element y if

- $x \sqsubseteq y$ and
- for every $z \sqsupseteq y$ and proposition $P : \mathcal{U}$, we have $(z = \bigvee \delta_{x,z,P}) \Rightarrow P$.

Definition

A $\delta_{\mathcal{U}}$-complete poset X is *positive* if we have $x, y : X$ such that x is strictly below y.

Examples of nontriviality and positivity

Slogan

Positivity is to nontriviality what inhabitedness is to nonemptiness.

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Example

In the powerset, $\emptyset \neq A$ if and only if A is a nonempty subset, but $\emptyset$ is strictly below A if and only if A is an inhabited subset.

Example

In the type of propositions, $\mathbf{0} \neq P$ if and only if $\neg\neg P$ holds, but $\mathbf{0}$ is strictly below P if and only if P holds.

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  $\Omega_{\mathcal{U}}$ and $\Omega_{\mathcal{U}}^{\neg\neg}$ are small
- ✓ Various kinds of posets: $\delta_{\mathcal{U}}$-complete posets
- ✓ Trivial: positivity and nontriviality
- Small poset

(Locally) small $\delta_{\mathcal{U}}$-complete posets

Definition

A $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ is *locally small* if the truth-value $x \sqsubseteq y$ is small for every $x, y : X$.

Example

Our running examples $\Omega_{\mathcal{U}}$ and $\mathcal{P}(X)$ for $X : \mathcal{U}$ are locally small, as are the large dcpos in the Scott model of PCF.

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Example

Our running examples $\Omega_{\mathcal{U}}$ and $\mathcal{P}(X)$ for $X : \mathcal{U}$ are locally small, as are the large dcpos in the Scott model of PCF.

Definition

A $\delta_{\mathcal{U}}$-complete poset is *small* if it is locally small and its carrier is small.

Key results

Theorem

There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing holds.

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}$-Resizing holds.

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Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}\text{-Resizing}$ holds.

Therefore, without resizing, there are no small nontrivial dcpos.

These are theorems of HoTT/UF. We do not make reference to models.

Proof sketch: using retracts

Definition

For a $\delta_{\mathcal{U}}$-complete poset X with points $x \sqsubseteq y$, we define

$$\begin{aligned} \Delta_{x,y} : \Omega_{\mathcal{U}} &\to X \\ P &\mapsto \bigvee \delta_{x,y,P} \end{aligned}$$

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Lemma

A locally small $\delta_{\mathcal{U}}$-complete poset X with points $x \sqsubseteq y$ is nontrivial if and only if the composite $\Omega_{\mathcal{U}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{U}} \xrightarrow{\Delta_{x,y}} X$ is a section.

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Lemma

A locally small $\delta_{\mathcal{U}}$-complete poset X with points $x \sqsubseteq y$ is positive if and only if for every $z \sqsupseteq y$, the map $\Omega_{\mathcal{U}} \xrightarrow{\Delta_{x,z}} X$ is a section.

Back to the key results

Lemma

If $s : A \to B$ is a section and B is a small set, then A is small too.

Theorem

There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing holds.

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}$-Resizing holds.

Therefore, without resizing, there are no small nontrivial dcpos.

Decidable equality and excluded middle

Lemma

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Constructively and predicatively, (locally small) $\delta_{\mathcal{U}}$-complete posets cannot have decidable equality and are necessarily large.

Theorem

There is a locally small nontrivial $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if weak excluded middle in $\mathcal{U}$ holds.

Theorem

There is a locally small positive $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if excluded middle in $\mathcal{U}$ holds.

Conclusion

Take-home message

- Nontrivial/positive sup-lattices, dcpos, bounded-complete posets, etc., can only be small if (weak) impredicativity is assumed.
- Predicatively, universe level management is necessary.
  In particular, the dcpos in the Scott model of PCF are necessarily large.
- Nontrivial/positive locally small sup-lattices, dcpos, etc., can only have decidable equality if (weak) excluded middle is assumed.

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- Nontrivial/positive locally small sup-lattices, dcpos, etc., can only have decidable equality if (weak) excluded middle is assumed.

Further results in our paper

- Various fixed point theorems crucially rely on impredicativity.
- Zorn's Lemma implies impredicativity (but not excluded middle).
- Compare completeness with respect to subsets/families.