Set-Theoretic and Type-Theoretic Ordinals Coincide

Tom de Jong

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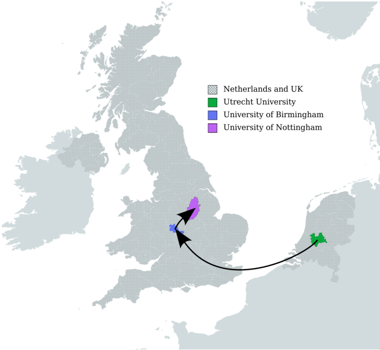
Faculty of Science
Postdoctoral and Fellow Research Conference

20 July 2023

Who am I?

- 2012–2015: Joint BSc in *Computer Science* and *Mathematics*, Utrecht University (Netherlands).
- 2015–2018: MSc in *Mathematical Sciences*, Utrecht University.
- 2018–2022: PhD in *Theoretical Computer Science*, University of Birmingham.
- Since Oct 2022: Research Fellow, *Functional Programming Lab*, School of Computer Science.



Our publication

- Our paper was accepted to the Logic in Computer Science (LICS) conference, which is the most prestigious and competitive publication venue in our field.
- We worked on mathematical logic and produced computer verified proofs ensuring the correctness of our arguments.

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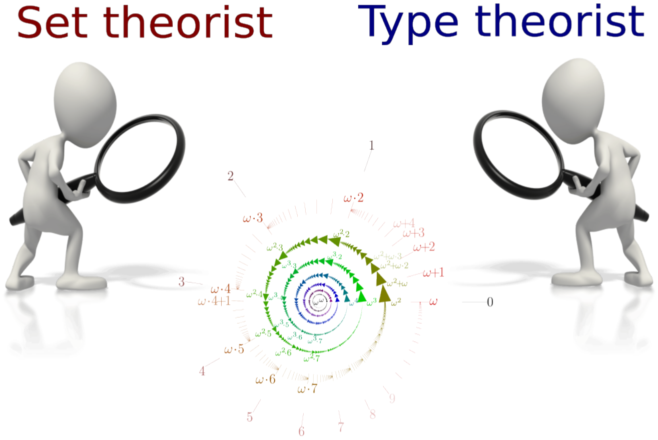
- We worked on mathematical logic and produced computer verified proofs ensuring the correctness of our arguments.
- I had a leading role in this project:
  - I largely wrote the first half of the paper and the accompanying computer checked proofs.
  - My questions and framing started the investigations that led to the second half of the paper.
  - I acted as project manager, which included setting deadlines and making sure we met them.

Our work in a nutshell

We showed that two *different* mathematical foundations, namely set theory and type theory, agree on the notion of ordinal number, a fundamental concept in mathematical logic and computer science.



Set theory

- The default foundation for all of mathematics.
- Rich history going back to the 1870s.
- Syntactically, it is minimalistic: one only needs logic and a *single* relation $\in$ used to encode everything else.

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$$\tau(x) := \forall y \forall z (z \in y \Rightarrow y \in x \Rightarrow z \in x)$$

$$\tau(x) \land \forall y (y \in x \Rightarrow \tau(y))$$

Type theory

- Martin-Löf's dependent type theory is an alternative foundation for mathematics, originally developed in the 1960s.
- Recent (2010s) insights from topology and higher category theory led to homotopy type theory.

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- Recent (2010s) insights from topology and higher category theory led to homotopy type theory.
- Implemented in proof assistants such as Agda which allows us computer verify our constructions and proofs.

```
An ordinal is a type equipped with ordinal structure.

\begin{code}

OrdinalStructure : 𝓤 ̇ → 𝓤 ⁺ ̇

OrdinalStructure {𝓤} X = Σ _<_ : (X → X → 𝓤 ̇) , (is-well-order _<_)

is-well-order : 𝓤 ⊔ 𝓥 ̇

is-well-order = is-prop-valued
              × is-well-founded
              × is-extensional
              × is-transitive
```

What are ordinal numbers?

One answer: numbers for transfinite counting/ordering.

$$0,\ 1,\ 2,\ 3,\ \ldots\ \omega,\ \omega+1,\ \omega+2,\ \ldots\ \omega\cdot 2+19, \ldots$$

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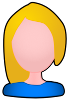

Alice
Sells & builds houses

Bob
Wants to buy

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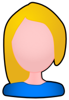
When is my house ready?



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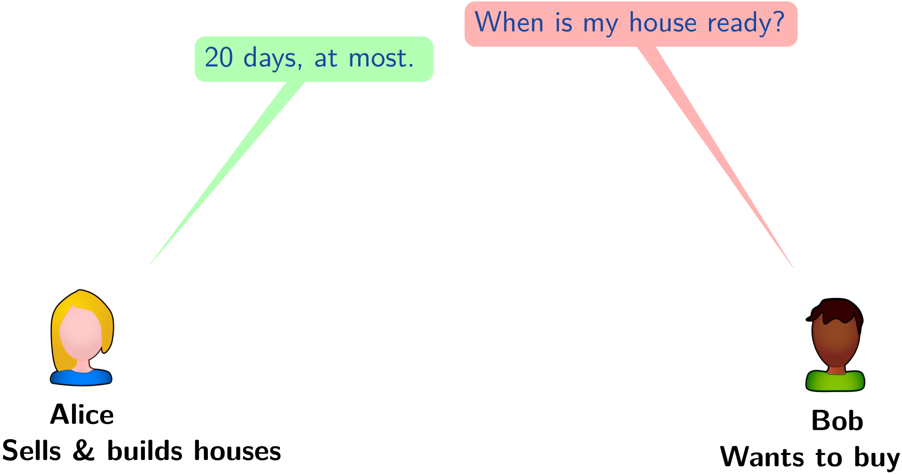

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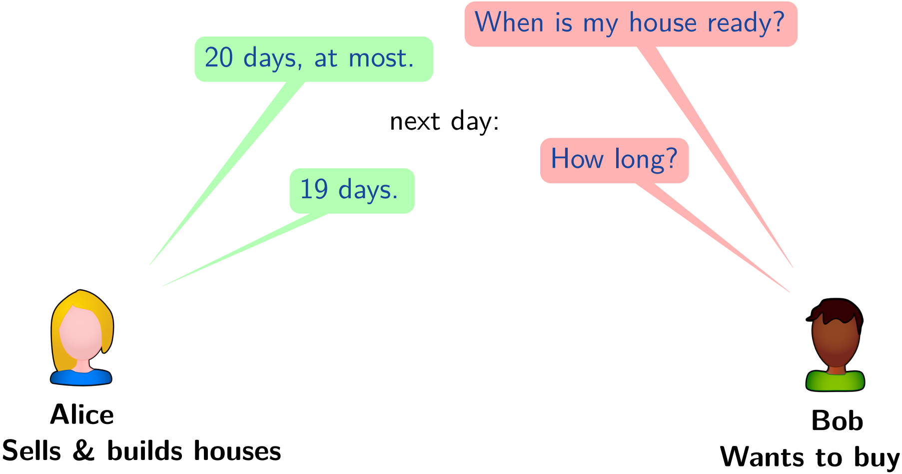

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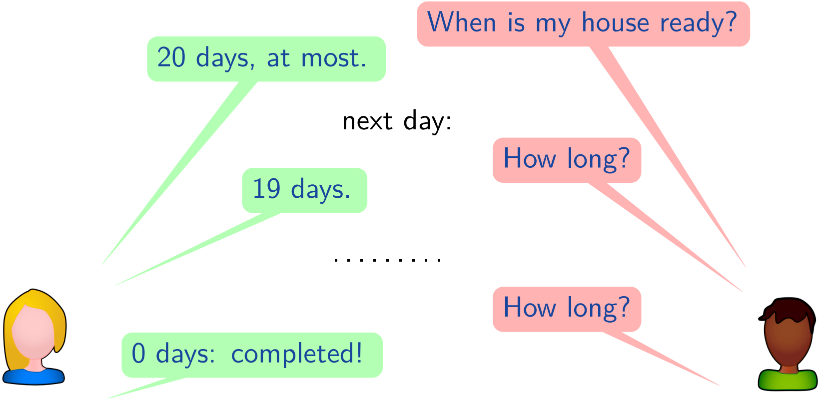

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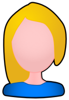

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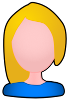
ω+1 days.



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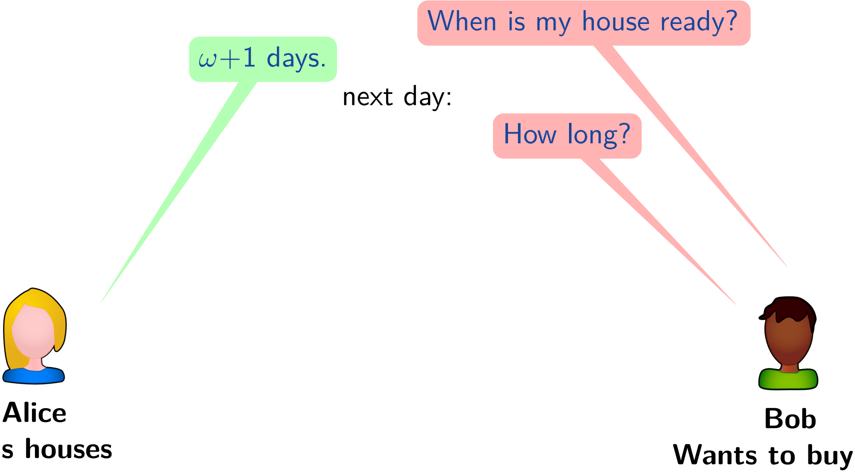

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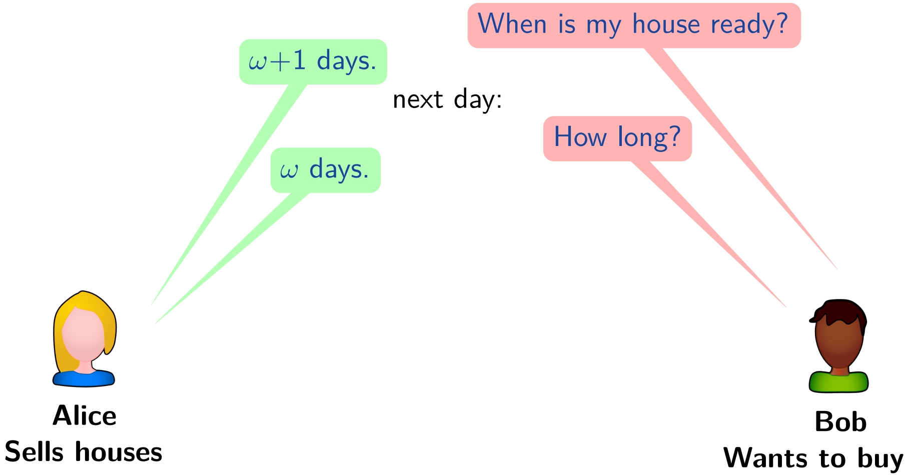

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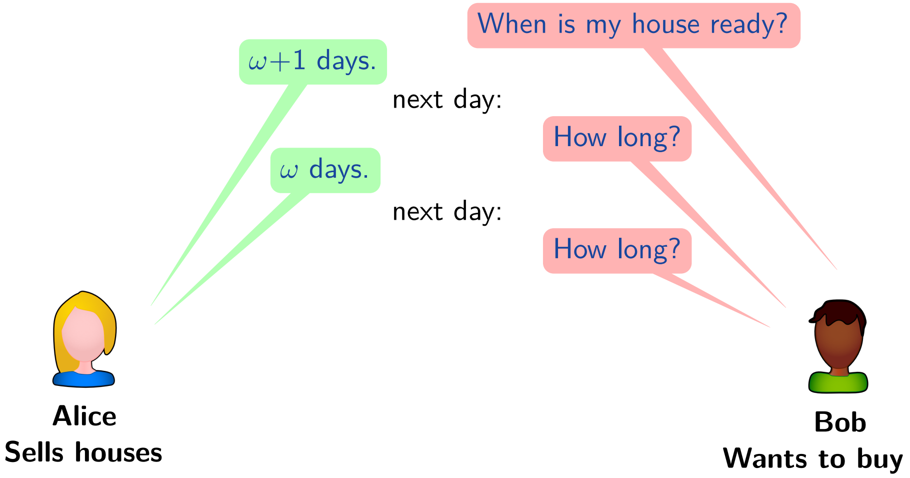

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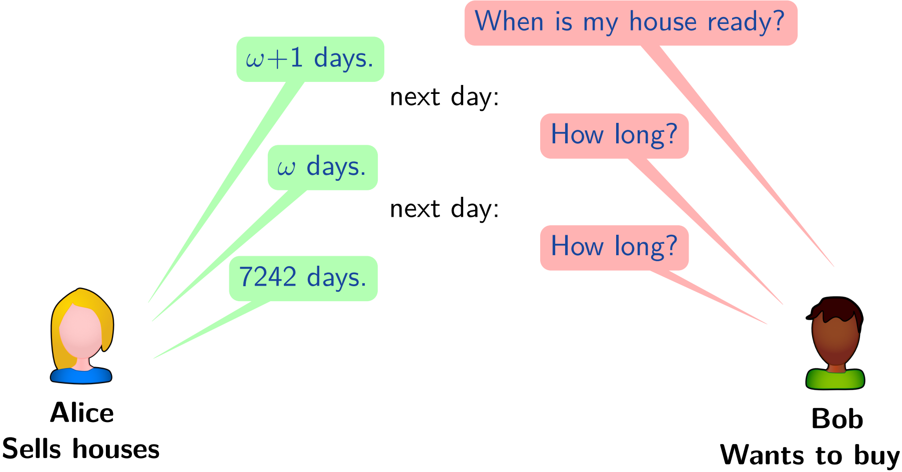

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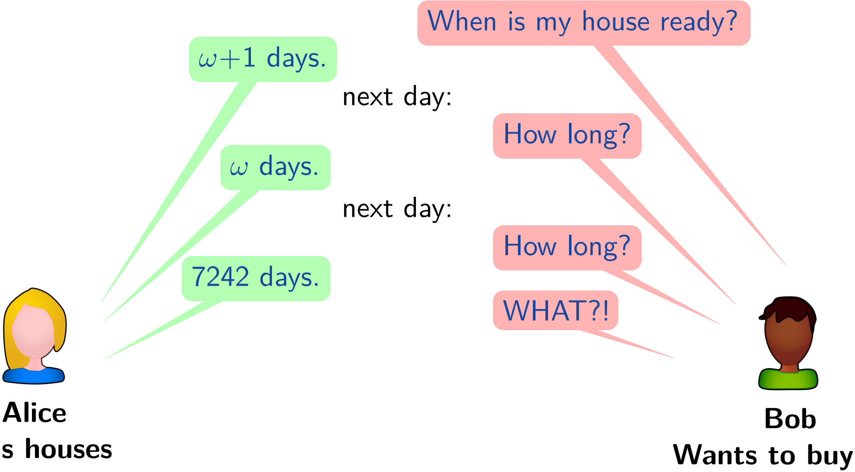

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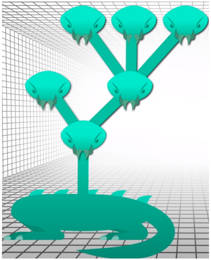

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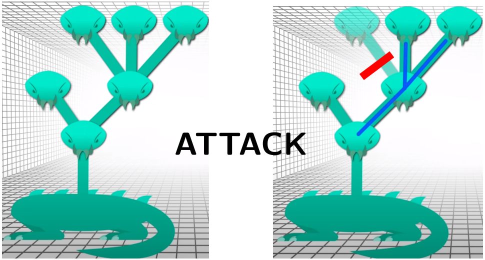

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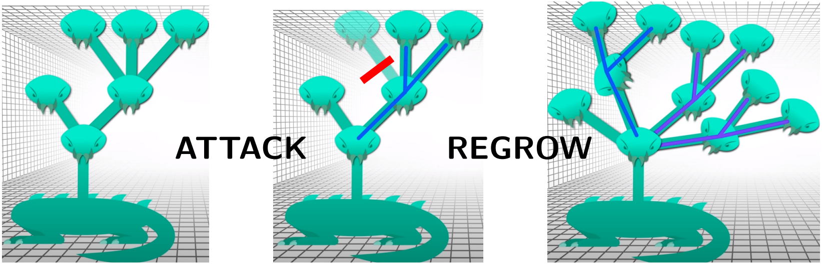

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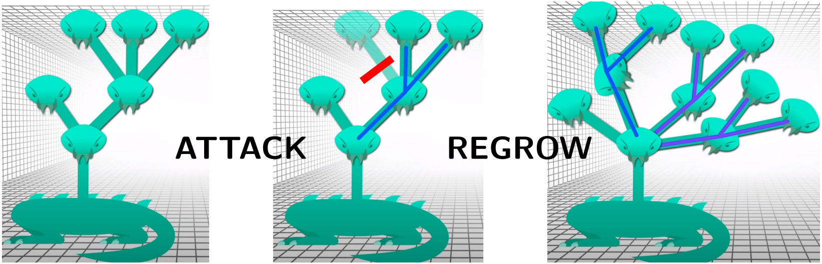

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Can we kill the Hydra?

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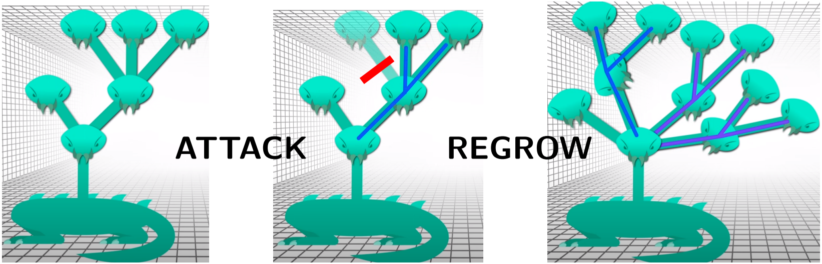

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Can we kill the Hydra?

Using ordinals we can *prove* that the Hydra can be beaten!

Conclusion

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