Synthetic homotopy theory at work: epimorphisms, suspensions and groups

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Notes for an invited talk on 24 June 2026 at the workshop
Formal proof and synthetic mathematics

1. Preliminaries

I am very grateful to the organizers, both for putting on the workshop and for the honourable invitation. I feel that we are at an exciting point in time where, thanks to proof assistants, logical foundations are more relevant to (mainstream) mathematics than ever. It is a privilege to be given the opportunity to contribute to what I consider to be an important development by participating in this workshop.

Rather than presenting results, my goal is to showcase typical *ideas* and *methods*. How do we develop (synthetic) mathematics in HoTT?

This is based on a paper with Ulrik Buchholtz and Egbert Rijke: Epimorphisms and Acyclic Types in Univalent Foundations. In: The Journal of Symbolic Logic (2025), pp. 1–36. doi: 10.1017/jsl.2024.76.

Related work

There is related work on acyclic spaces and epimorphisms of spaces.

The following two papers work with "CW-spaces", i.e. topological spaces having the homotopy type of a CW-complex.

- HAUSMANN, Jean-Claude and HUSEMÖLLER, Dale, 1979. Acyclic maps. *L'enseignement Mathématique.* 1979. Vol. 25, no. 1–2, p. 53–75. DOI 10.5169/seals-50372.
- ALONSO, Juan M., 1983. Fibrations that are cofibrations. *Proceedings of the American Mathematical Society.* 1983. Vol. 87, no. 4, p. 749–753. DOI 10.1090/s0002-9939-1983-0687656-3.

Raptis works in the ∞-topos of spaces and makes use of Whitehead's principle / hypercompleteness (∞-connected $\Rightarrow$ contractible)

- RAPTIS, George, 2019. Some characterizations of acyclic maps. *Journal of Homotopy and Related Structures.* 2019. Vol. 14, no. 3, p. 773–785. DOI 10.1007/s40062-019-00231-6.

Hoyois works with general ∞-toposes, but makes use of their site presentations

- HOYOIS, Marc, 2019. *On Quillen's plus construction.* Online. Available from: https://hoyois.app.uni-regensburg.de/papers/acyclic.pdf

In contrast, our work is fully internal and can be interpreted in arbitrary ∞-toposes, cf. SHULMAN, Michael, 2019. *All $(\infty, 1)$-toposes have strict univalent universes.* Online. Available from: https://arxiv.org/abs/1904.07004

2. Technical preliminaries I

Truncated types

A type A is

- contractible if it has exactly one element, i.e. we have an element of

$$\sum(a : A), \prod(b : A), a = b.$$

  The only example is the unit type $\mathbb{1}$ but contractible types are nonetheless extremely important!
- a subsingleton (or proposition) if it has at most one element, i.e. we have $\prod(a, b : A), a = b$.

- a set if all its identity types are subsingletons (i.e. it has no higher identifications).
  Examples include most familiar types, e.g. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, etc.

We can define 1-groupoids as those types whose identity types are sets, and continue to get a hierarchy:
contractible types ⊂ subsingletons ⊂ sets ⊂ 1-groupoids ⊂ 2-groupoids ⊂ …

In homotopy theory we shift by -2, so this hierarchy becomes
-2-truncated types ⊂ -1-truncated types ⊂ 0-truncated types ⊂ …
Often we shorten "k-truncated type" to "k-type".

Prime example of a 1-type: the (homotopy) circle S^1. It is freely generated (a "higher inductive type" — a HIT) generated by a point pt and a loop at that point. We have $(\mathrm{pt} = \mathrm{pt}) \simeq \mathbb{Z}$.

Truncations

We assume to have operations $\| - \|_k$ that reflect (in the categorical sense of obeying a suitable universal property) a type into a k-type.

For $k = -1$, we omit the subscript. This is the propositional truncation.
We write $\exists(a : A), P(a)$ as a shorthand for $\| \sum(a : A), P(a) \|$.
We say A is inhabited if we have an element of $\|A\|$; if we have an element of A, then A is pointed. The latter is *structure*!

k-connectedness

A type A is k-connected if its k-truncation is contractible. *Idea*: the type has no interesting structure below level k. *Example*: the circle S^1 is 0-connected.

Note: -1-connected is the same as inhabited.

We say that a type is ∞-connected if it is k-connected for all k. Such types need not be contractible: this assumption is Whitehead's principle / hypercompleteness.

We fiberwise extend the notion of k-connectedness to maps.

Fibers

The fiber of a map $f : A \to B$ at b is the type $\sum(a : A), f(a) = b$.

A map is
- an equivalence if all of its fibers are contractible. (Equivalently, it has a section and a retraction.)
- a surjection if all of its fibers are inhabited. Spelled out, $\prod(b : B), \exists(a : A), f(a) = b$.
  Note that an element of $\prod(b : B), \sum(a : A), f(a) = b$ would give a *section* of f instead.
- an embedding if all of its fibers are subsingletons. Equivalently, the canonical map

  $$a = b \to f(a) = f(b)$$

  is an equivalence for all $a, b : A$. Compare this to the definition of an injection where we merely ask that the implication $f(a) = f(b) \to a = b$ holds.

NB. All these notions define subsingleton types, i.e. properties and not structure. In particular, two equivalences/surjections/embeddings are equal exactly when they are equal as maps.

3. Epimorphisms

In 1-category theory, we say that $f : A \to B$ is an epi(morphism) whenever it holds that for every two arrows $g, h : B \to X$, if $g \circ f = h \circ f$ then $g = h$.

Definition. A map $f : A \to B$ is an epi(morphism) if precomposition by f is an embedding, i.e. if the canonical map $g = h \to g \circ f = h \circ f$ is an equivalence for every $g, h : B \to X$.

(Compare with embeddings and injections.)

Observation. A map $f : A \to B$ is an epi if and only if the type of extensions of $k : A \to X$ along f has at most one element, i.e. the extension is unique if it exists.
Indeed, given $k : A \to X$, the fiber of f^* (precomposition by f) is $\sum(e : B \to X), e \circ f = k$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k}\downarrow & \circlearrowleft \quad \swarrow {\scriptstyle e} & \\ X & & \end{array}$$

Note that the identification is part of the data!

Non-example. For $\mathbb{2} := \mathbb{1} + \mathbb{1}$, the map $f : \mathbb{2} \to \mathbb{1}$ is **not** an epimorphism.
Proof: Easy calculation! Consider $X := S^1$ and take $k : \mathbb{2} \to S^1$ to be $[\mathrm{pt}, \mathrm{pt}]$, and note that

$$\begin{aligned} \sum(e : \mathbb{1} \to S^1), e \circ f = k \quad &\simeq \quad \underbrace{\sum(x : S^1), (x = \mathrm{pt})}_{\text{contractible!}} \quad \times (x = \mathrm{pt}) \\ &\simeq (\mathrm{pt} = \mathrm{pt}) \\ &\simeq \mathbb{Z}, \end{aligned}$$

This is HoTT's version of the Yoneda lemma in terms of significance (they are also related/analogous)

so that there are $\mathbb{Z}$ many extensions! ■

A priori, our notion of epimorphism is not universe independent: there is a quantification over "all" maps. Fortunately, we can show the following, which should be familiar from 1-category theory.

Proposition. A map $f : A \to B$ is an epi if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array}$$

is a pushout, i.e. the codiagonal $\nabla_f : B +_A B \to B$ is an equivalence.

Proof. The square is a pushout if and only if the diagonal map

$$\begin{aligned} (B \to X) &\to \sum(g : B \to X), \sum(h : B \to X), g \circ f = h \circ f \\ k &\mapsto (k, k, \mathrm{refl}) \end{aligned}$$

is an equivalence. But the RHS is $\sum(g : B \to X), \mathrm{fib}_{f^*}(g \circ f)$, so this happens iff $\mathrm{fib}_{f^*}(g \circ f)$ is contractible. Since it is always inhabited, we just need it to be a subsingleton, which happens iff f^* is an embedding. ■

4. Acyclic types

Classically, a space is said to be acyclic ("no cycles") if its reduced homology vanishes.
This notion comes up in K-theory.

We prefer the following definition and refer to the traditional definition as "homology acyclic".
Choosing suitable definitions for HoTT is part of the challenge!

Definition. A type A is acyclic if its suspension is contractible.

Definition. The suspension ΣA of a type is the pushout of $\mathbb{1} \leftarrow A \rightarrow \mathbb{1}$. [Draw picture]

This pushout is (automatically) like a homotopy pushout as everything is up to the identity type.

Example. The suspension of $\mathbb{2}$ is the circle S^1. Higher spheres are defined via suspensions: $S^{n+1} :\equiv \Sigma S^n$.

Proposition. Acyclic implies homology acyclic. With Whitehead's principle (= every ∞-connected type is contractible), the converse holds too.

Theorem. A map is epi if and only if all of its fibers are acyclic types.

Proof. This amounts to the claim that the codiagonal is the fiberwise suspension, i.e. $\mathrm{fib}_{\nabla_f}(b) \simeq \Sigma\, \mathrm{fib}_f(b)$. This follows from the "flattening lemma" (sometimes called "descent"), i.e. that we can pull back a pushout to obtain another pushout. Indeed, since $\mathrm{fib}_{f(b)}$ is the pullback of f along $b : \mathbb{1} \to B$, we can pull back the pushout square of the codiagonal along b to get the desired result. [Draw diagrams] ■

It is high time for some examples of acyclic types (and hence of epimorphisms), but I will postpone them until the end of the talk!

5. Interlude: groups as certain types

For a pointed type $a : A$, the type $\Omega A := (a = a)$ has a group structure: the operation is path (really, loop) concatenation and refl is the neutral element.

Every pointed type thus yields a group: $A \mapsto \|\Omega A\|_0$. This is the fundamental group $\pi_1(A)$. If we restrict ourselves to 1-types, we can just consider ΩA, i.e. without set truncating. If we restrict to 0-connected types, then we've ensured that the choice of base point is irrelevant. In fact, we may identify groups with 0-connected, pointed 1-types, and group homomorphisms with pointed maps between such types.
If A is a 0-connected, pointed 1-type such that $\Omega A \simeq G$ for a given group G, then we say that A is a delooping of G, and that A is the classifying type of G, and sometimes write A as $\mathrm{B}G$.
Example: $\mathrm{B}\mathbb{Z} = S^1$.

For more, see

- BUCHHOLTZ, Ulrik, DOORN, Floris van and RIJKE, Egbert, 2018. Higher Groups in Homotopy Type Theory. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. Association for Computing Machinery. 2018. p. 205–214. LICS '18. DOI 10.1145/3209108.3209150.
- BEZEM, Marc, BUCHHOLTZ, Ulrik, CAGNE, Pierre, DUNDAS, Bjørn Ian and GRAYSON, Daniel R., 2026. *Symmetry*. Online. Available from: https://github.com/UniMath/SymmetryBook

The Eckmann–Hilton argument says: higher loop spaces are commutative, hence the group structure on $\pi_{k+1}(A) := \|\Omega^k A\|_0$ is commutative for $k \geq 2$.

This is powerful and we can extend the above correspondence to *abelian* groups. We can identify abelian groups with 1-connected, pointed 2-types!

6. k-acyclic types and group theory [if time permits (unlikely)]

We can consider a relative version of epimorphism.

Definition. Say that $f : A \to B$ is a k-epi for $k \in \{-2, -1, 0, ...\}$ if $f^* : (B \to X) \to (A \to X)$ is an embedding for all k-types X.

Definition. A type A is k-acyclic if its suspension is k-connected, i.e. $\|\Sigma A\|_k$ is contractible.

Note: every acyclic type is k-acyclic for any k.

Theorem. The following are equivalent for a map $f : A \to B$:

1. f is a k-epi;
2. $\|f\|_k : \|A\|_k \to \|B\|_k$ is a k-epi;

3. the codiagonal ∇_f is k-connected;
4. the fibers of f are all k-acyclic.

Observations.

- 0-acyclic = −1-connected. This recovers that epis of sets are precisely the surjections.
- k-connected ⇒ $(k+1)$-acyclic because suspensions increase connectedness by 1. Shameless plug:

  JONG, Tom de, 2025. Formalizing Equivalences Without Tears. In: MØGELBERG, Rasmus Ejlers and BERG, Benno van den (eds..), *30th International Conference on Types for Proofs and Programs (TYPES 2024)*. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. 2025. p. 1:1–1:6. Leibniz International Proceedings in Informatics (LIPIcs). DOI 10.4230/LIPIcs.TYPES.2024.1.

Specializing to $A \to \mathbb{1}$, we obtain via some straightforward work the following result.
Corollary. The following are equivalent:
1. the type A is k-acyclic,
2. for all k-types B, the constants map $B \to (A \to B)$ is an embedding,
3. for all k-types B and $x, y : B$, the constants map $x = y \to (A \to x = y)$ is an equivalence.

Note how the third item is equivalent to the second.

Theorem. For a *set* A, we have

$$A \text{ is 1-acyclic} \iff A \text{ is acyclic} \iff A \text{ is contractible.}$$

Hence, there are no acyclic *sets*. We must look at higher types.

Proof. It suffices to prove that every 1-acyclic set is contractible. Let G be the free group on the 1-acyclic set A. We consider the 1-type $\mathrm{B}G$. By the Corollary, the constants map $\mathrm{pt} = \mathrm{pt} \to (A \to \mathrm{pt} = \mathrm{pt})$ is an equivalence, i.e. the map $x \mapsto \_ \mapsto x$ is an equivalence $G \to (A \to G)$. Hence, every map $A \to G$ must be constant. But this includes the inclusion of generators $A \hookrightarrow G$, so that all elements of A must be equal. Finally, A is 1-acyclic and hence 0-acyclic and hence −1-connected, so inhabited. Thus, A is contractible. ■

Corollary. 1-acyclic ⟺ 0-connected.

Corollary. The following are equivalent for a group homomorphism $f : G \to H$:
1. f is an epi of groups;
2. $\mathrm{B}f : \mathrm{B}G \underset{\mathrm{pt}}{\to} \mathrm{B}H$ is an epi of pointed connected 1-types;
3. $\mathrm{B}f : \mathrm{B}G \to \mathrm{B}H$ is a 0-connected map of types;
4. f is surjective as a map of sets.

Many traditional proofs of the fact that the epimorphisms of groups are precisely the surjections rely on excluded middle. For instance, the suggested proof in Mac Lane [Exercise I.5.5] relies heavily on a case analysis that requires the law of excluded middle. A notable exception is Todd Trimble's proof which is constructive and uses group actions.

- TRIMBLE, Todd, 2020. *epimorphisms of groups are surjective*. Online. Available from: https://ncatlab.org/nlab/show/epimorphisms+of+groups+are+surjective

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Usually, a group G is said to be *perfect* if it equals its own commutator subgroup G'. Since the abelianization of G is given by G/G', we reformulate perfectness as follows.

Definition. A group G is perfect if its abelianization is trivial.

Example: The alternating group A_5.

We recall from the Symmetry book that abelianization as a map

$$(\text{groups} \to \text{abelian groups})$$
$$\text{pointed 0-connected 1-types} \to \text{pointed 1-connected 2-types}$$

is given by $\mathrm{B}G \mapsto \|\Sigma\,\mathrm{B}G\|_2$.

This yields:
Theorem. The classifying type $\mathrm{B}G$ of a group G is 2-acyclic if and only if G is perfect.
More generally, a type A is 2-acyclic if and only if A is 0-connected and $\pi_1(A, a)$ is perfect for every $a : A$.

7. Examples of acyclic types

Hatcher's complex

We import Hatcher's 2-dimensional complex (Example 2.38 in Hatcher) as a HIT X with constructors

$$\mathrm{pt} : X, \quad a, b : \Omega X, \quad r : a^5 = b^3, \quad s : b^3 = (ab)^2.$$

Naturally, Hatcher gives a homological proof of acyclicity. Instead, we give a type theoretic proof using Eckmann–Hilton.

Definition.
- A Hatcher structure on a pointed type A is given by identifications

$$a, b : \Omega A, \quad r : a^5 = b^3, \quad s : b^3 = (ab)^2.$$

- A Hatcher algebra is a pointed type equipped with Hatcher structure.
- Note: X is just the initial Hatcher algebra.

Lemma. Every loop space, pointed at refl, has a unique Hatcher structure.

Proof. The type of Hatcher structures on a loop space ΩA is

$$\begin{aligned}
&\sum (a, b : \Omega^2 A), (a^5 = b^3) \times (b^3 = (ab)^2) \\
&\simeq [...] \times (b = a^2) && \text{(by Eckmann–Hilton)} \\
&\simeq \sum (a : \Omega^2)(a^5 = a^6) && \text{(by contracting)} \\
&\simeq \sum (a : \Omega^2)(a = \mathrm{refl}) && \text{(by cancelling } a^5\text{)} \\
&\simeq \mathbb{1}.
\end{aligned}$$

■

Proposition. The type X is acyclic.

Proof. For all pointed types Y, we have

$$\left(\Sigma X \underset{\mathrm{pt}}{\to} Y\right) \underbrace{\simeq}_{\substack{\text{loop space} \\ \text{suspension adjunction}}} \left(X \underset{\mathrm{pt}}{\to} \Omega Y\right) \underbrace{\simeq}_{\text{by universal prop. of } X} \text{Hatcher-structure}(\Omega Y) \underbrace{\simeq}_{\text{by the lemma}} \mathbb{1}.$$

Hence, ΣX has the universal property of $\mathbb{1}$ as a *pointed* type, so it must be contractible. ■

Proposition. The type X is nontrivial. More specifically, we have a 0-connected map from X to $\mathrm{B}A_5$ and hence a surjection $\pi_1(X) \twoheadrightarrow A_5$ to the alternating group on 5 generators.

Proof. The assignment $\mathrm{pt} \mapsto \mathrm{pt}; a \mapsto (12345); b \mapsto (254)$ does the job.

Higman's group

Higman's group H is given by the presentation

$$H = \langle a, b, c, d \mid a = [d, a], b = [a, b], c = [b, c], d = [c, d] \rangle,$$

where $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$.

Similar presentations with ≤ 3 generators are all trivial. The fact that H is in fact a nontrivial group is usually proved via some combinatorial group theory and lemmas about HNN-extensions and amalgamations.

I will sketch a proof via **descent** (= in a commutative cube where the top and bottom squares are pushouts, the front faces are pullbacks if the back faces are) and David Wärn's **zigzag construction** instead. This is a genuinely new construction in higher category theory.

- WÄRN, David, 2024. *Path spaces of pushouts*. Online. Available from: https://arxiv.org/abs/2402.12339

The presentation complex is easily imported into HoTT as a HIT with a single point constructor pt and four path constructors corresponding to the generators a, b, c, d and four 2-cells corresponding to the relations.
We will denote it by $\mathrm{B}H$; the notation will be justified momentarily.

Using Eckmann–Hilton again as above, one shows:
Proposition. $\mathrm{B}H$ is acyclic.

Theorem (∗). The type $\mathrm{B}H$ is a 1-type and the generators a, b, c, d all have infinite order.

The proof relies on the following result of Wärn.

Theorem (Wärn). Given a span $A \leftarrow R \rightarrow B$ of 0-truncated maps of 1-types, its pushout $A +_R B$ is again a 1-type and the inclusion maps are 0-truncated.
A k-truncated map is just a map whose fibers are all k-truncated.

Proof sketch of (∗). Re-express $\mathrm{B}H$ via iterated pushouts:

$$\begin{array}{ccc} \mathrm{B}\langle b\rangle & \longrightarrow & \mathrm{B}\langle b,c\rangle \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{B}\langle b,c\rangle & \longrightarrow & \mathrm{B}\langle a,b,c\rangle \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{B}\langle a,c\rangle & \longrightarrow & \mathrm{B}\langle a,b,c\rangle \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{B}\langle c,d,a\rangle & \longrightarrow & \mathrm{B}H \end{array}$$

Here, each type is the HIT that uses *only* those constructors of $\mathrm{B}H$ that involve the mentioned generators.

Idea: use Wärn's theorem to prove that all types are 1-types and that all maps are 0-truncated.
In particular, the composite $\mathrm{B}\langle b\rangle \to \mathrm{B}\langle b,c\rangle \to \mathrm{B}\langle a,b,c\rangle \to \mathrm{B}H$ is then 0-truncated, which yields a -1-truncated map, i.e. an embedding, at the level of loop spaces, i.e. an embedding $\langle b\rangle \hookrightarrow H$, so that b has infinite order. (Similar arguments apply for the other generators.)

Carrying out this idea requires descent at one point; I refer to the paper for the full details. ■

8. Future work

In higher topos theory, Raptis and Hoyois showed that the acyclic maps are the left class in an **orthogonal factorization system** (a **modality**).
The right class is given by the fiberwise hypoabelian maps. (A type X is hypoabelian if every perfect subgroup of $\pi_1(X, x)$ is trivial, for every $x : X$.)

If we had this, then some closure properties that we prove manually (in the paper) would follow from the general theory of modalities.

An essential ingredient in constructing the modality is Quillen's plus-construction (from K-theory).

In spaces, the modality is accessible, i.e. it can be described as nullification at a small collection of spaces.

Can we establish the acyclic maps as an (accessible) modality in HoTT? Could it be independent from HoTT?

Even without the construction of the factorization system, we can consider the hypoabelian maps, but the following orthogonality result seems to require a new principle.

Proposition (PP). The acyclic maps are left orthogonal to the hypoabelian maps.

(PP) Plus Principle. Every 1-connected acyclic type is contractible.

Via the Freudenthal suspension theorem and a cute iterative argument we can prove:
Proposition. Every 1-connected acyclic type is ∞-connected.
Hence, (PP) follows from hypercompleteness / Whitehead's principle.
The canonical countermodel for hypercompleteness, the ∞-topos of parameterized spectra, *does* validate (PP) though (as shown by Mathieu Anel).

The **Kan–Thurston theorem** says that an ∞-group can be presented by a pair (G, P) of a group G and a perfect normal subgroup $P \lhd G$. Can we do this type-theoretically?

- KAN, Daniel M. and THURSTON, William P., 1976. Every connected space has the homology of a $K(\pi, 1)$. *Topology*. 1976. Vol. 15, no. 3, p. 253–258. DOI 10.1016/0040-9383(76)90040-9.