On epimorphisms and acyclic types in HoTT

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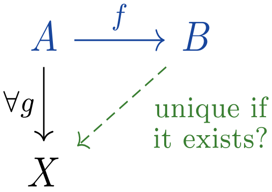
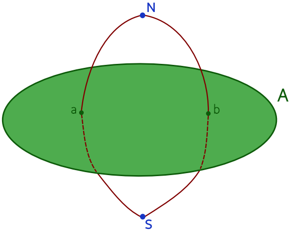
Second International Conference on
Homotopy Type Theory (HoTT)

Carnegie Mellon University, Pittsburgh, USA

25 May 2023

Epimorphisms and acyclic types





- We develop the synthetic homotopy theory of acyclic types.
  Classically, acyclic spaces are used in
  - Quillen's plus construction,
  - the Kan–Thurston theorem, and
  - the Barratt–Priddy(–Quillen) theorem.
- We turn to algebraic topology to answer a question about (potentially higher) types:
  What are the epimorphisms of types?

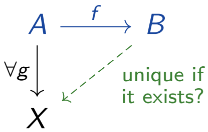
Epimorphisms

- In 1-category theory, a map $f : A \to B$ is an epi(morphism) if for every $g, h : B \to C$ we have

$$g \circ f = h \circ f \implies g = h.$$

  In other words, $(-) \circ f$ is an injection.

- Note:



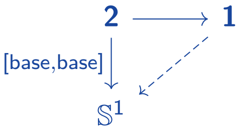
$\iff$ f is an epi

- Def. A map $f : A \to B$ is an epi if $(-) \circ f$ is an embedding.

(Non)examples of epimorphisms

- While the map $\mathbf{2} \to \mathbf{1}$ is surjective and an epi of sets, it is *not* an epi of types.

  It is not an epi, because the type of (dashed) extensions



  is not a proposition, as it is equivalent to

$$\sum_{x:\mathbb{S}^1} (x = \text{base}) \times (x = \text{base}) \;\simeq\; (\text{base} = \text{base}) \;\simeq\; \mathbb{Z}.$$

- Nontrivial examples of epis will be presented later.

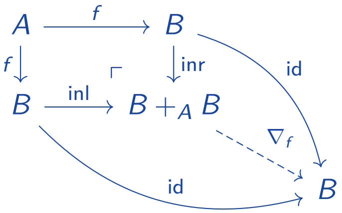
Epimorphisms and pushouts

- Lemma A map $f : A \to B$ is an epi if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array}$$

  is a pushout.

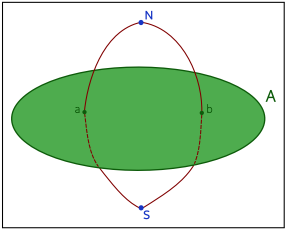
- Cor. A map $f : A \to B$ is epic if and only if its codiagonal ∇_f is an equivalence.



Acyclic types and maps

- Def. The suspension ΣA of a type A is the pushout

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow N \\ \mathbf{1} & \underset{S}{\longrightarrow} & \Sigma A \end{array}$$



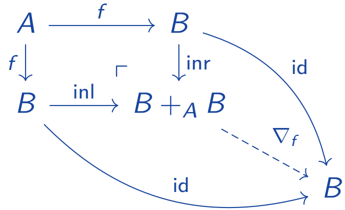
- Def. A type A is acyclic if its suspension ΣA is contractible.

- Def. A map is acyclic if all its fibers are.

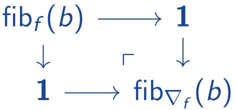
Acyclicity and codiagonals

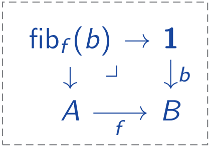
- Lemma The codiagonal is the fiberwise suspension:
  if $f : A \to B$, then $\mathrm{fib}_{\nabla_f}(b) \simeq \Sigma\, \mathrm{fib}_f(b)$.

- Proof. By descent we can pull back the pushout square



along $1 \xrightarrow{b} B$ to get the *pushout* square





which is the defining pushout for the suspension. □

The epimorphisms are the acyclic maps

- Thm. A map is an epi if and only if it is acylic.

- Proof. $f : A \to B$ is an epi

$$\iff \begin{array}{ccc} A & \xrightarrow{f} & B \\ f\downarrow & & \downarrow\text{id} \\ B & \xrightarrow[\text{id}]{} & B \end{array} \text{ is a pushout}$$

$\iff$ $\nabla_f : B +_A B \to B$ is an equivalence

$\iff$ $\mathrm{fib}_{\nabla_f}(b)$ is contractible for all $b : B$

$\iff$ $\Sigma\,\mathrm{fib}_f(b)$ is contractible for all $b : B$

$\iff$ f is acyclic. $\square$

Closure properties

- Every equivalence is an epi.
- The epis (equivalently, acyclic maps) satisfy a 3-for-2 property: for f acyclic, the composite $g \circ f$ is acyclic if and only if g is.
- Epis are stable under pushouts along arbitrary maps.
- Thanks to the theorem, being epic is a fiberwise notion. Thus, epis are stable under pullbacks and retracts.
- It also follows that epimorphisms satisfy the precomposition-embedding property for dependent maps: Precomposition by $f : A \to B$ is an embedding

$$\prod_{b:B} P(b) \xhookrightarrow{(-)\circ f} \prod_{a:A} P(f(a))$$

for all $P : B \to \mathsf{Type}$.

No acyclic sets

- Thm. A set is acyclic if and only if it is contractible.
- Thus, discard sets when looking for interesting acyclic types.

No acyclic sets

- Thm. A set is acyclic if and only if it is contractible.

- Proof. Let G be the free group on an acyclic set A with inclusion of generators $\eta : A \hookrightarrow G$. If A is acyclic, then $A \to \mathbf{1}$ is an epi, so the constant map

$$\mathrm{B}G \to (A \to \mathrm{B}G) \qquad x \mapsto \lambda(a : A).x$$

is an embedding. Hence, the constant map $G \to (A \to G)$ is an equivalence. Thus,

$$\eta(x) = \eta(y) \qquad \forall\, x, y : A.$$

But η is also an embedding (†), so A must be a subsingleton. Finally, A is also inhabited, because it is acyclic. □

- (†) This was shown constructively by Mines, Richman and Ruitenburg; formalized in Agda by Escardó (j.w.w. Bezem, Coquand and Dybjer) and a new synthetic proof was recently given by Wärn.

Relation to connectedness

- Prop. Every acyclic type is (0-)connected.
- Prop. Every 1-connected (i.e. simply connected) acyclic type is ∞-connected.

  Thus, assuming Whitehead's Principle, every 1-connected acyclic type is contractible.
- So we should turn to 0-connected types for acyclicity.

Relation to connectedness

- Prop. Every 1-connected (i.e. simply connected) acyclic type is ∞-connected.

  Thus, assuming Whitehead's Principle, every 1-connected acyclic type is contractible.

- Proof. By the Freudenthal suspension theorem, the unit $\sigma : A \to \Omega\Sigma A$ of the loop-suspension adjunction is $2n$-connected whenever A is n-connected (for $n \geq 0$).

  If A is acyclic, then $\Sigma A \simeq \mathbf{1}$ and $\Omega\Sigma A \simeq \mathbf{1}$, so the connectedness of σ is that of A.

  Now if A is also 1-connected, then σ, and hence A, is in turn 2-connected, then 4-connected, etc., hence 2^n-connected for all n. □

First example of a nontrivial acyclic type

- A non-trivial example of an acyclic space can by found in Hatcher's textbook (Ex. 2.38).

- In HoTT, we can import this as a HIT X with constructors:

$$\begin{aligned} \mathrm{pt} &: X \\ a, b &: \mathrm{pt} = \mathrm{pt} \\ r &: a^5 = b^3 \\ s &: b^3 = (ab)^2 \end{aligned}$$

- Why is X nontrivial?

- Why is X acyclic?

Nontriviality of Hatcher's example

- Definition of X as a HIT:

$$\mathsf{pt} : X \qquad a, b : \mathsf{pt} = \mathsf{pt} \qquad r : a^5 = b^3 \qquad s : b^3 = (ab)^2$$

- We define a map from X to the classifying type $\mathsf{B}A_5$ of the alternating group on 5 elements:

  On paths, this is defined by

$$\begin{aligned} (\mathsf{pt} =_X \mathsf{pt}) &\to A_5 \\ a &\mapsto (12345) \\ b &\mapsto (254) \end{aligned}$$

  which can be shown to respect the relations r and s.

- These cycles generate A_5, so the map on paths is surjective. Hence, X must be nontrivial.

Acyclicity of Hatcher's example

- Definition of X as a HIT:

$$\mathsf{pt} : X \qquad a, b : \mathsf{pt} = \mathsf{pt} \qquad r : a^5 = b^3 \qquad s : b^3 = (ab)^2$$

- We study the suspension ΣX as a HIT and simplify:

$$\begin{array}{|l|} \hline N, S : \Sigma X \\ m_{\mathsf{pt}} : N = S \\ \vdots \\ \\ \hline \end{array}$$

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- We study the suspension ΣX as a HIT and simplify:

$$\boxed{\begin{array}{l} N, S : \Sigma X \\ m_{\mathrm{pt}} : N = S \\ \quad \vdots \end{array}} \xrightarrow{\text{Contr. at } (N, m_{\mathrm{pt}})} \boxed{\begin{array}{rl} N & : \Sigma X \\ m_a, m_b & : \mathrm{refl}_N = \mathrm{refl}_N \\ m_r & : m_a^5 = m_b^3 \\ m_s & : m_b^3 = (m_a\, m_b)^2 \end{array}}$$

Acyclicity of Hatcher's example (continued)

- The crux is that higher homotopy groups are abelian by the Eckmann–Hilton (EH) argument.

$$
\begin{aligned}
N &: \Sigma X \\
m_a, m_b &: \mathsf{refl}_N = \mathsf{refl}_N \\
m_r &: m_a^5 = m_b^3 \\
m_s &: m_b^3 = (m_a\, m_b)^2
\end{aligned}
$$

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- And we can contract again:

$$\boxed{\begin{array}{c} N : \Sigma X \\ m_a, m_b : \mathsf{refl}_N = \mathsf{refl}_N \\ m_r : m_a^5 = m_b^3 \\ m_s : m_b = m_a^2 \end{array}} \overset{\text{Contr. at } (m_b, m_s)}{\rightsquigarrow} \boxed{\begin{array}{c} N : \Sigma X \\ m_a : \mathsf{refl}_N = \mathsf{refl}_N \\ m_r : m_a^5 = m_a^6 \end{array}}$$

Acyclicity of Hatcher's example (continued)

- We can simplify

$$\boxed{\begin{array}{l} N : \Sigma X \\ m_a : \mathsf{refl}_N = \mathsf{refl}_N \\ m_r : m_a^5 = m_a^6 \end{array}} \rightsquigarrow \boxed{\begin{array}{l} N : \Sigma X \\ m_a : \mathsf{refl}_N = \mathsf{refl}_N \\ m_r : m_a = \mathsf{refl}_{\mathsf{refl}_N} \end{array}}$$

  and finally, contract once again:

$$\boxed{\begin{array}{l} N : \Sigma X \\ m_a : \mathsf{refl}_N = \mathsf{refl}_N \\ m_r : m_a = \mathsf{refl}_{\mathsf{refl}_N} \end{array}} \overset{\text{Contr. at } (m_a, m_r)}{\rightsquigarrow} \boxed{N : \Sigma X}$$

- Thus, the suspension ΣX of X is equivalent to a single point and X is acyclic.

The Higman group: an acyclic classifying type

- The Higman group is defined as the group with 4 generators a, b, c, d and 4 relations

  $$r_a : a = [d, a] \quad r_b : b = [a, b] \quad r_c : c = [b, c] \quad r_d : d = [c, d],$$

  where $[x, y] \equiv xyx^{-1}y^{-1}$ denotes the commutator.

- In HoTT we can describe its classifying type $\mathrm{B}H$ as a HIT:

$$\begin{aligned} \mathrm{pt} &: \mathrm{B}H \\ a, b, c, d &: \mathrm{pt} = \mathrm{pt} \\ r_a &: a = [d, a] \\ r_b &: a = [a, b] \\ r_c &: a = [b, c] \\ r_d &: a = [c, d] \end{aligned}$$

Acyclicity and nontriviality of the Higman HIT

- The commutators become trivial in the suspension by Eckmann–Hilton, so as with Hatcher's example, the type $\mathrm{B}H$ is seen to be acyclic.
- Why is $\mathrm{B}H$ a nontrivial type?

  For $n \leq 3$ generators and relations, the resulting group turns out to be trivial!
- Classical proofs of the nontriviality of Higman's group rely on combinatorial group theory and show that all generators a, b, c, d have infinite order in H.

Nontriviality of the Higman group via descent and pushouts

- We can *completely avoid* classical combinatorial group theory using descent and David Wärn's recent results on identity types of pushouts.

- Thm. (Wärn) Given a pushout square

$$\begin{array}{ccc} R & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{inr}} \\ A & \xrightarrow[\mathsf{inl}]{} & A +_R B \end{array}$$

with f and g 0-truncated maps of 1-types, the pushout $A +_R B$ is again a 1-type and inl and inr are 0-truncated.

- We can (re)construct $\mathsf{B}H$ as a series of such pushout squares.

- It also follows that $\mathsf{B}H$ is a 1-type: no need to truncate!

Summary

At higher types, the notion of epimorphism

- becomes quite strong,
- coincides with the notion of an acyclic map, and
- is interesting from the p.o.v. of synthetic homotopy theory.

Additional and future work

- Do the acyclic maps form an accessible modality?

- Many properties seem to need an additional axiom:

  > Plus Principle: Every acyclic and simply connected type is contractible.

  It follows from Whitehead's Principle (WP) and was highlighted by Hoyois in ∞-topos theory.

- We believe that plus-constructions can be performed in HoTT assuming WP, Sets Cover, and Countable Choice.

- Use the theory of binate groups to prove acyclicity of some infinitely presented groups?

- We also study k-epimorphisms and k-acyclic types. (Similar to k-equivalences and k-connected maps.)

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