Domain theory in predicative Univalent Foundations

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Introduction

Overarching goal

Develop domain theory constructively and predicatively in Univalent Foundations.

Domain theory is a branch of order theory with applications in:

- semantics of programming languages;
- topology and algebra;
- higher-type computability.

See for instance [Str06; Sco72] for applications of domain theory to the semantics of programming languages and [AJ94; Sco72; Gie+80; Gie+03] for applications of domain theory in topology and algebra. See [LN15] for higher-type computability.

You can develop large parts of domain theory predicatively, e.g.

- Scott's model of PCF [Plo77; Sco93; Jon19];
- Scott's model D_∞ of the untyped λ-calculus [Sco72];
- continuous and algebraic dcpos, (abstract) bases and (rounded) ideal completion [AJ94; Gie+80; Gie+03].

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- semantics of programming languages;
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- higher-type computability.

In previous work we have given a predicative account of:

- Scott's model of PCF and its computational adequacy;
- Scott's model D_∞ of the untyped λ-calculus;
- continuous and algebraic dcpos

See for instance [Str06; Sco72] for applications of domain theory to the semantics of programming languages and [AJ94; Sco72; Gie+80; Gie+03] for applications of domain theory in topology and algebra. See [LN15] for higher-type computability.

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This talk

Study impredicativity in domain-theoretic context. Examples of domain-theoretic statements that imply impredicativity:

- having small domains;
- Zorn's lemma;
- Pataraia's lemma.

In this talk, we will instead focus on what *cannot* be done predicatively. We will identify some domain-theoretic statements that are inherently impredicative, by showing that they imply resizing principles.

We hope that this will increase our understanding of not just domain theory, but of impredicativity in Univalent Foundations too.

We will explain what we mean by "impredicativity", before moving on to the domain theory part of this talk.

Impredicativity

- Voevodsky introduced resizing rules for propositions in UF.
- We consider resizing axioms instead.
  - The axioms follow from excluded middle and so are consistent (by the simplicial sets model).
  - We prove things about the axioms internally, i.e. within the theory, rather than the meta-theory.
- Predicativity is the absence of such resizing axioms.

See [Voe11; Voe15] for the resizing rules that Voevodsky proposed.

See [KL18] for the simplicial sets model.

Examples of resizing axioms

Definition

If $X : \mathcal{U}$ is a type and $\mathcal{V}$ is a universe, then we define:

$$X \text{ has-size } \mathcal{V} :\equiv \sum_{Y:\mathcal{V}} (Y \simeq X).$$

Again, note that these principles follow from excluded middle (for a universe $\mathcal{U}$), since it says that every proposition $P : \mathcal{U}$ is equivalent to $\mathbf{0}$ or $\mathbf{1}$. And we have equivalent copies of those in any universe $\mathcal{V}$.

Notice that (X has-size $\mathcal{V}$) is a proposition if and only if we have univalence.

A computational interpretation of these axioms is an open problem [Swa19; Uem19].

See [Uni13, Section 3.5] and [Esc20] for more on impredicativity in UF.

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Resizing axiom ($\Omega_{\mathcal{U}}$-impredicativity)

Let U be a type universe. $\Omega_{\mathcal{U}}$-impredicativity asserts that the type $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \text{is-prop}(P)$ has size $\mathcal{U}$.

Resizing axiom (Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$)

Let $\mathcal{U}, \mathcal{V}$ be universes. Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$ asserts that every proposition in $\mathcal{U}$ has size $\mathcal{V}$.

Again, note that these principles follow from excluded middle (for a universe $\mathcal{U}$), since it says that every proposition $P : \mathcal{U}$ is equivalent to $\mathbf{0}$ or $\mathbf{1}$. And we have equivalent copies of those in any universe $\mathcal{V}$.

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Motivating (counter)examples

Classically, a directed complete poset (dcpo) is a poset such that every directed subset has a supremum.

Naive type-theoretic definition

A dcpo is a set $P : \mathcal{U}$ with a partial order $- \sqsubseteq - : P \to P \to \mathcal{U}$ such that every directed family $I \to P$ (with $I : \mathcal{U}$) has a supremum.

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Constructive issue

Suppose that $\mathbf{2}$ with $0 \sqsubseteq 1$ is a dcpo. Let P be a proposition. Then the directed family $1 + P \to \mathbf{2}$, $\mathsf{inl}(\star) \mapsto 0$, $\mathsf{inr}(p) \mapsto 1$ has a supremum. It is either 0 or 1, but this gives $\neg P + \neg\neg P$.

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Predicative issue

The poset $\Omega_{\mathcal{U}_0}$ ordered by a implication has suprema of (directed) families $Q_{(-)} : I \to \Omega_{\mathcal{U}_0}$ given by $\exists_{i:I} Q_i$ as long as $I : \mathcal{U}_0$.
But $\Omega_{\mathcal{U}_0} : \mathcal{U}_1$ does not have suprema when $I : \mathcal{U}_1$ unless propositional resizing holds.

Basic objects in domain theory

Definition

Let $(P, \sqsubseteq)$ be a poset. A family $u : I \to P$ is *directed* if I is inhabited and $\Pi_{i,j:I}\exists_{k:I}(u_i \sqsubseteq u_k \times u_j \sqsubseteq u_k)$.

For predicativity reasons (see previous slide), we define dcpos using families, rather than subsets and we index the notion by a "small" universe.

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Fix a universe $\mathcal{V}$ of "small" types.

Definition

A *$\mathcal{V}$-directed complete poset* (*$\mathcal{V}$-dcpo*) is a poset $(P, \sqsubseteq)$ such that every directed family $I \to P$ with $I : \mathcal{V}$ has a least upper bound $\bigsqcup \alpha$ in $(P, \sqsubseteq)$.

Definition

A $\mathcal{V}$-dcpo is called *pointed* if it has a least element $\bot$.

For predicativity reasons (see previous slide), we define dcpos using families, rather than subsets and we index the notion by a "small" universe.

Large dcpos with suprema of small directed families

Typical examples of dcpos $(D, \sqsubseteq)$ will have:

- suprema of directed families indexed by types in $\mathcal{U}_0$;
- the order $\sqsubseteq$ taking values in $\mathcal{U}_0$ (up to equivalence);
- the carrier D in $\mathcal{U}_1$.

This should be compared to category theory. One has large categories with small (co)limits (even impredicatively), cf. [Shu11].

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E.g.

- $\Omega_{\mathcal{U}_0}$;
- Scott's D_∞;
- rounded ideal completion of dyadics rationals $n/2^m$ in the interval $(-1, 1)$.

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Predicatively, are there ($\mathcal{U}_0$-)dcpos in the first universe $\mathcal{U}_0$?

This should be compared to category theory. One has large categories with small (co)limits (even impredicatively), cf. [Shu11].

Can dcpos be small?

Definition

A $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ is *small* if D has size $\mathcal{V}$.

Definition

A pointed $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ is *non-trivial* if $\exists_{x:D}\, x \neq \bot$.

We generalize the question from the previous slide and answer it negatively.

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Theorem (I)

If we have a non-trivial small pointed $\mathcal{V}$-dcpo, then the type $(\Omega_{\neg\neg})_{\mathcal{V}} :\equiv \sum_{P:\Omega_{\mathcal{V}}} (\neg\neg P \to P)$ of $\neg\neg$-stable propositions in $\mathcal{V}$ has size $\mathcal{V}$.

Take-home message

Predicatively, dcpos are necessarily large.

We generalize the question from the previous slide and answer it negatively.

Can we get full impredicativity?

Non-triviality is rather weak constructively. There is the (constructively) stronger notion of having a positive element.

Theorem (I')

If we have a small pointed $\mathcal{V}$-dcpo with a positive element, then we have $\Omega_{\mathcal{U}}$-impredicativity.

See [Joh84] for the definition of a *positive* element: $x : D$ is *positive* if whenever P is a proposition and $\alpha : P \to D$ is such that its supremum $\bigvee \alpha$ exists, then the inequality $x \sqsubseteq \bigvee \alpha$ implies P.

Proof (sketch) of Theorem I

Lemma

If X is a retract of Y and the section is an embedding, then Y has-size $\mathcal{V} \to X$ has-size $\mathcal{V}$.

Proof.

If $r : Y \to X$ has a section s that is an embedding, then $\left(\sum_{y:Y} \|s(r(y)) = y\|\right) \simeq X$ via $(y, p) \mapsto r(y)$. □

The corollary follows, since every left-cancellable map into a set is an embedding.

Strictly speaking the map $x \mapsto x(x \neq \bot)$ is not well-defined, since $x \neq \bot$ is only small up to equivalence. But this is easily fixed, while it is more convenient to describe the map as if D is strictly small.

Combining the two lemmas now gives the theorem.

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Corollary

If X is a retract of a set Y, then Y has-size $\mathcal{V} \to X$ has-size $\mathcal{V}$.

Lemma

If $(D, \sqsubseteq)$ is a non-trivial small pointed $\mathcal{V}$-dcpo, then the map $x : D \mapsto (x \neq \bot) : (\Omega_{\neg\neg})_{\mathcal{V}}$ has a section.

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Zorn's lemma

Definition

An element m in a poset $(P, \sqsubseteq)$ is *maximal* if for every $x : P$, the inequality $m \sqsubseteq x$ implies $x = m$.

It should be noted that [Bel97]:

- Zorn's lemma does not imply excluded middle;
- (hence) Zorn's lemma does not imply the axiom of choice in the absence of excluded middle.

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Zorn's lemma (variation)

Let $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{T}$ be universes. Zorn's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ says that every pointed $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ with $D : \mathcal{U}$ and $\sqsubseteq$ taking values in $\mathcal{T}$, has a maximal element.

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Theorem (II)

Let $\mathcal{V}$ and $\mathcal{U}$ be universes. Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

E.g. Zorn's-Lemma$_{\mathcal{U}_0,\mathcal{U}_1,\mathcal{U}_0}$ *implies* Propositional-Resizing$_{\mathcal{U}_1,\mathcal{U}_0}$.

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Proof overview

- Fix universes $\mathcal{V}$ and $\mathcal{U}$ and assume Zorn's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$.
- Let $P : \mathcal{U}$ be any proposition. We must show that P has size $\mathcal{V}$.
- Consider the type of "small" propositions that imply P: $\mathcal{L}_{\mathcal{V}}(P) :\equiv \sum_{Q:\mathcal{V}} \text{is-prop}(Q) \times (Q \to P) : \mathcal{V}^+ \sqcup \mathcal{U}$.

Lemma

With the implication ordering, $\mathcal{L}_{\mathcal{V}}(P)$ is a pointed $\mathcal{V}$-dcpo.

- By assumption, it has a maximal element, a proposition $R : \mathcal{V}$ with $R \to P$.
- But $P \to R$ must also hold. For if P is inhabited, then $\mathcal{L}_{\mathcal{V}}(P) \simeq \mathcal{L}_{\mathcal{V}}(\mathbf{1}) \simeq \Omega_{\mathcal{V}}$, so that by maximality, $R = \mathbf{1}$.
- Hence, $R \simeq P$, so P has size $\mathcal{V}$.

The least element of $\mathcal{L}_{\mathcal{V}}(P)$ is given by:

$$(\mathbf{0}, \mathbf{0}\text{-is-prop}, \text{unique-from-}\mathbf{0}).$$

See [EK17] for more on the lifting $\mathcal{L}$.

Pataraia's lemma

Pataraia's fixed point theorem

Let $(D, \sqsubseteq)$ be a pointed dcpo. Then every monotone endomap on D has a least fixed point.

Pataraia's proof of this theorem is constructive, but impredicative (it uses "large" suprema).

See [Esc03, Section 2] for a constructive, but impredicative proof of Pataraia's fixed point theorem [Pat97].

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Let $(D, \sqsubseteq)$ be a pointed dcpo. Then every monotone endomap on D has a least fixed point.

Pataraia's proof of this theorem is constructive, but impredicative (it uses "large" suprema).
The key ingredient is Pataraia's lemma: every dcpo has a greatest monotone inflationary endomap.

Definition

Let $(P, \sqsubseteq)$ be a poset. An endomap $f : P \to P$ is *inflationary* if $x \sqsubseteq f(x)$ for every $x : P$.

See [Esc03, Section 2] for a constructive, but impredicative proof of Pataraia's fixed point theorem [Pat97].

Pataraia's lemma

Let $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{T}$ be universes. Pataraia's-Lemma$_{\mathcal{V},\mathcal{U},\mathcal{T}}$ asserts that every $\mathcal{V}$-dcpo $(D, \sqsubseteq)$ with $D : \mathcal{U}$ and $\sqsubseteq$ taking values in $\mathcal{T}$, has a greatest monotone inflationary endomap.

Theorem (III)

Let $\mathcal{V}$ and $\mathcal{U}$ be universes. Pataraia's-Lemma$_{\mathcal{V},\mathcal{V}^+ \sqcup \mathcal{U},\mathcal{V}}$ *implies* Propositional-Resizing$_{\mathcal{U},\mathcal{V}}$.

E.g. Pataraia's-Lemma$_{\mathcal{U}_0,\mathcal{U}_1,\mathcal{U}_0}$ *implies* Propositional-Resizing$_{\mathcal{U}_1,\mathcal{U}_0}$.

Since the proof of Pataraia's theorem is constructive, it is provable if we have propositional resizing.

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E.g. Pataraia's-Lemma$_{\mathcal{U}_0,\mathcal{U}_1,\mathcal{U}_0}$ *implies* Propositional-Resizing$_{\mathcal{U}_1,\mathcal{U}_0}$.

Ironically, Pataraia's-Lemma$_{\mathcal{U},\mathcal{U},\mathcal{U}}$ is a theorem even in our predicative framework, but by Theorem I there are no interesting examples to apply it to, unless propositional resizing holds.

Since the proof of Pataraia's theorem is constructive, it is provable if we have propositional resizing.

Proof overview

- Fix universes $\mathcal{V}$ and $\mathcal{U}$ and assume Pataraia's-Lemma$_{\mathcal{V},\mathcal{V}^+\sqcup\mathcal{U},\mathcal{V}}$.
- Let $P : \mathcal{U}$ be any proposition. We must show that P has size $\mathcal{V}$.
- As before, consider the pointed $\mathcal{V}$-dcpo of "small" propositions that imply P:
  $\mathcal{L}_{\mathcal{V}}(P) :\equiv \sum_{Q:\mathcal{V}} \text{is-prop}(Q) \times (Q \to P) : \mathcal{V}^+ \sqcup \mathcal{U}$.
- By assumption, it has a greatest monotone inflationary endomap $g : \mathcal{L}_{\mathcal{V}}(P) \to \mathcal{L}_{\mathcal{V}}(P)$.
- Applying g to $\mathbf{0}$, we get a proposition $R : \mathcal{V}$ such that $R \to P$.
- But $P \to R$ must also hold. For if P is inhabited, then $\mathcal{L}_{\mathcal{V}}(P) \simeq \mathcal{L}_{\mathcal{V}}(\mathbf{1}) \simeq \Omega_{\mathcal{V}}$. Since g is the greatest monotone inflationary endomap, $R = \mathbf{1}$ follows.
- Hence, $R \simeq P$, so P has size $\mathcal{V}$.

Conclusion and future work

Some natural order-theoretic statements (e.g. Zorn's Lemma) imply resizing axioms. And predicatively, dcpos are necessarily large.

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Typically, our dcpos have large carriers, are locally small and have suprema of small directed families.

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Typically, our dcpos have large carriers, are locally small and have suprema of small directed families.

Future work

- What about Pataraia's fixed point *theorem*? Does it imply some resizing axiom? Or is there an alternative predicative proof?
- We have a predicative account of continuous and algebraic dcpos. What about exponentials of such dcpos? Do SFP domains or bounded complete dcpos work nicely predicatively?

References I

[AJ94] S. Abramsky and A. Jung. 'Domain Theory'. In: *Handbook of Logic in Computer Science*. Ed. by S. Abramsky, D. M. Gabbay and T. S. E. Maibaum. Vol. 3. Updated online version available at: `https://www.cs.bham.ac.uk/~axj/pub/papers/handy1.pdf`. Clarendon Press, 1994, pp. 1–168.

[Bel97] J. L. Bell. 'Zorn's Lemma and Complete Boolean Algebras in Intuitionistic Type Theories'. In: *The Journal of Symbolic Logic* 62.4 (1997). DOI: 10.2307/2275642, pp. 1265–1279.

[Esc03] Martín H. Escardó. 'Joins in the Frame of Nuclei'. In: *Applied Categorical Structures* 11 (Apr. 2003). DOI: 10.1023/A:1023555514029, pp. 117–124.

References II

[EK17] Martín H. Escardó and Cory M. Knapp. 'Partial Elements and Recursion via Dominances in Univalent Type Theory'. In: *26th EACSL Annual Conference on Computer Science Logic (CSL 2017)*. Ed. by Valentin Goranko and Mads Dam. Vol. 82. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2017, 21:1–21:16. DOI: 10.4230/LIPIcs.CSL.2017.21.

[Esc20] Martín Hötzel Escardó. '*Introduction to Univalent Foundations of Mathematics with Agda*'. arXiv: 1911.00580 [cs.LO]. Feb. 2020.

References III

[Gie+03] G. Gierz et al. *Continuous Lattices and Domains*. Vol. 93. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2003. DOI: 10.1017/CBO9780511542725.

[Gie+80] Gerhard Gierz et al. *A Compendium of Continuous Lattices*. Springer Berlin Heidelberg, 1980. DOI: 10.1007/978-3-642-67678-9.

[Joh84] Peter T. Johnstone. 'Open locales and exponentiation'. In: *Mathematical Applications of Category Theory*. Ed. by J. W. Gray. Vol. 30. Contemporary Mathematics. American Mathematical Society, 1984, pp. 84–116. DOI: 10.1090/conm/030/749770.

References IV

[Jon19] Tom de Jong. *The Scott model of PCF in univalent type theory*. Nov. 2019. arXiv: 1904.09810 [math.LO].

[KL18] Chris Kapulkin and Peter LeFanu Lumsdaine. *The Simplicial Model of Univalent Foundations (after Voevodsky)*. Oct. 2018. arXiv: 1211.2851 [math.LO].

[LN15] John Longley and Dag Normann. *Higher-Order Computability*. Springer, 2015. DOI: 10.1007/978-3-662-47992-6.

[Pat97] Dito Pataraia. 'A constructive proof of Tarski's fixed-point theorem for dcpos'. Presented at the 65th Peripatetic Seminar on Sheaves and Logic, Aarhus, Denmark. Nov. 1997.

References V

[Plo77] G. D. Plotkin. 'LCF considered as a programming language'. In: *Theoretical Computer Science* 5.3 (1977), pp. 223–255. DOI: 10.1016/0304-3975(77)90044-5.

[Sco72] Dana Scott. 'Continuous lattices'. In: *Toposes, Algebraic Geometry and Logic*. Ed. by F.W. Lawvere. Vol. 274. Lecture Notes in Mathematics. DOI: 10.1007/BFB0073967. Springer, 1972, pp. 97–136.

[Sco93] Dana S. Scott. 'A type-theoretical alternative to ISWIM, CUCH, OWHY'. In: *Theoretical Computer Science* 121.1 (1993), pp. 411–440. DOI: 10.1016/0304-3975(93)90095-B.

References VI

[Shu11] Mike Shulman. *Topos Theory Can Make You a Predicativist*. Blog post. Jan. 2011. URL: https://golem.ph.utexas.edu/category/2011/01/topos_theory_can_make_you_a_pr.html.

[Str06] Thomas Streicher. *Domain-Theoretic Foundations of Functional Programming*. DOI: 10.1142/6284. World Scientific, 2006.

[Swa19] Andrew W. Swan. 'Choice, Collection and Covering in Cubical Sets'. Talk in the electronic *HoTTEST* seminar, 6 November. Slides at https://www.uwo.ca/math/faculty/kapulkin/seminars/hottestfiles/Swan-2019-11-06-HoTTEST.pdf. Video recording at https://www.youtube.com/watch?v=r9KbEOzyr1g. 2019.

References VII

[Uem19] Taichi Uemura. 'Cubical Assemblies, a Univalent and Impredicative Universe and a Failure of Propositional Resizing'. In: *24th International Conference on Types for Proofs and Programs (TYPES 2018)*. Ed. by Peter Dybjer, José Espírito Santo and Luís Pinto. Vol. 130. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2019, 7:1–7:20. DOI: 10.4230/LIPIcs.TYPES.2018.7.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

References VIII

[Voe11] Vladimir Voevodsky. 'Resizing Rules — their use and semantic justification'. Slides from a talk at *TYPES*, Bergen, 11 September. Available at https://www.math.ias.edu/vladimir/sites/math.ias.edu.vladimir/files/2011_Bergen.pdf. 2011.

[Voe15] Vladimir Voevodsky. 'An experimental library of formalized mathematics based on the univalent foundations'. In: *Mathematical Structures in Computer Science* 25.5 (2015), pp. 1278–1294.