Order Theory and Propositional Resizing in HoTT/UF

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Introduction I

PhD research

Develop domain theory in constructive and predicative (i.e. without propositional resizing) HoTT/UF.

Domain theory

Domain theory is a branch of order theory with applications in:

- semantics of programming languages;
- topology and algebra;
- higher-type computability.

HoTT/UF

Sophisticated foundation for mathematics that is constructive by default.

Introduction II

Motivating observations

- In the Scott model of the programming language PCF in HoTT/UF, the directed complete posets (dcpos) interpreting PCF types are large.

  E.g. $\mathbb{N}$ is in $\mathcal{U}_0$, but $[\![\mathtt{nat}]\!]$ is in $\mathcal{U}_1$.
- In pointfree topology in HoTT/UF: the locales/frames are large.

Main point of this talk

We show that this largeness is unavoidable in HoTT/UF unless we assume propositional resizing.

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We show that this largeness is unavoidable in HoTT/UF unless we assume propositional resizing.

Based on our *FSCD'21* paper

de J. and Martín Hötzel Escardó, *Predicative Aspects of Order Theory in Univalent Foundations*, LIPIcs (195), 2021.
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Main result in this talk

Theorem (crude formulation)

Various kinds of nontrivial posets are small in HoTT/UF *if and only if propositional resizing holds.*

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Theorem (Freyd) for comparison

A category with small (co)limits is small if and only if it is a poset.

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Theorem (crude formulation)

Various kinds of nontrivial posets are small in HoTT/UF *if and only if propositional resizing holds.*

Items to be made precise

- Propositional resizing
- Various kinds of posets
- Nontrivial
- Small poset

The precise formulation will be a theorem of HoTT/UF. We do not make reference to models.

Propositional resizing

Definition

The type $\Omega_{\mathcal{U}} :\equiv \sum_{P:\mathcal{U}} \text{is-prop}(P)$ is the type of all propositions in a universe $\mathcal{U}$.

It's important to notice that $\Omega_{\mathcal{U}}$ lives in the next universe $\mathcal{U}^+$.

Definition

A type $X : \mathcal{U}^+$ is *small* if we have $Y : \mathcal{U}$ with $Y \simeq X$.

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Definition

The axiom $\Omega_{\mathcal{U}}\text{-Resizing}$ asserts that $\Omega_{\mathcal{U}}$ is small.

Open question

Does $\Omega_{\mathcal{U}}\text{-Resizing}$ have a computational interpretation, like univalence in cubical type theory?

Axioms vs rules

- Vladimir Voevodsky proposed a propositional resizing rule, i.e. instead of having a type in $\mathcal{U}$ that is *equivalent* to $\Omega_{\mathcal{U}}$, we postulate $\Omega_{\mathcal{U}} : \mathcal{U}$.
- We study propositional resizing axioms so that we can prove theorems about them inside HoTT/UF, rather than metatheorems.
- The rule is not known to be consistent, but the axiom is, because it follows from excluded middle which is validated by the simplicial sets model.

Excluded middle implies $\Omega_{\mathcal{U}}$-Resizing

Proposition

Excluded middle in $\mathcal{U}$ implies $\Omega_{\mathcal{U}}$-Resizing.

Proof.

With excluded middle in $\mathcal{U}$ we have $\Omega_{\mathcal{U}} \simeq \mathbf{2}$. □

So in studying $\Omega_{\mathcal{U}}$-Resizing we *must* work constructively, i.e. without excluded middle.

Weak excluded middle and propositional resizing

Definition

- A proposition P is *$\neg\neg$-stable* if $\neg\neg P$ implies P.
- The type $\Omega_{\mathcal{U}}^{\neg\neg}$ is the type of all $\neg\neg$-stable propositions in a universe $\mathcal{U}$.
- The axiom $\Omega_{\mathcal{U}}^{\neg\neg}\text{-Resizing}$ asserts that $\Omega_{\mathcal{U}}^{\neg\neg}$ is small.

Definition

Weak excluded middle holds in a universe $\mathcal{U}$ if for every proposition P in $\mathcal{U}$ either $\neg\neg P$ holds or $\neg P$ does.

Proposition

Weak excluded middle in $\mathcal{U}$ implies $\Omega_{\mathcal{U}}^{\neg\neg}\text{-Resizing}$.

Making the theorem precise

Theorem (crude formulation)

Various kinds of nontrivial posets are small in HoTT/UF *if and only if propositional resizing holds.*

Items to be made precise

- ✓ Propositional resizing: $\Omega_{\mathcal{U}}$-Resizing and $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing.
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Small-complete posets

Definition

A poset $(X, \sqsubseteq)$ is a *$\mathcal{U}$-sup-lattice* if every family $I \to X$ with $I : \mathcal{U}$ has a supremum in X.

The carrier X and the values of $\sqsubseteq$ are *not* required to be in $\mathcal{U}$ or even in the same universe.

Definition

A poset X is a *$\mathcal{U}$-dcpo* if every *directed* family $I \to X$ with $I : \mathcal{U}$ has a supremum in X.

Examples of $\mathcal{U}$-sup-lattices

Example

The powerset $\mathcal{P}(X) :\equiv X \to \Omega_{\mathcal{U}}$ of $X : \mathcal{U}$ is $\mathcal{U}$-sup-lattice.

- If $I : \mathcal{U}$ and $A_{(-)} : I \to \mathcal{P}(X)$, then its supremum is the subset $x \mapsto \exists_{i:I} A_i(x)$.
- Note: $\exists_{i:I} A_i(x) : \mathcal{U}$, but $\mathcal{P}(X) : \mathcal{U}^+$.

Example

The type $\Omega_{\mathcal{U}}$ is a $\mathcal{U}$-sup-lattice ordered by implication and with suprema given by existential quantification.

Examples of $\mathcal{U}$-dcpos

Example

For any set $X : \mathcal{U}$, the *lifting* $\mathcal{L}(X) :\equiv \sum_{P:\Omega_{\mathcal{U}}}(P \to X)$ of X is a $\mathcal{U}$-dcpo which lives in $\mathcal{U}^+$.

- Any element $x : X$ gives an element in $\mathcal{L}(X)$ by taking the proposition P to be $\mathbf{1}$.
- In general, P is the domain of definition of the partial element.

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In particular, for the Scott model of PCF:

- $[\![\mathtt{nat}]\!] \equiv \mathcal{L}(\mathbb{N})$ is a $\mathcal{U}_0$-dcpo in $\mathcal{U}_1$.
- $[\![\mathtt{nat} \Rightarrow \mathtt{nat}]\!]$ is the $\mathcal{U}_0$-dcpo of Scott continuous functions from $\mathcal{L}(\mathbb{N})$ to $\mathcal{L}(\mathbb{N})$, which lives in $\mathcal{U}_1$ again.

$\delta_{\mathcal{U}}$-completeness

Definition

A poset $(X, \sqsubseteq)$ is *$\delta_{\mathcal{U}}$-complete* if for every proposition $P : \mathcal{U}$ and elements $x \sqsubseteq y$, the family

$$\begin{aligned} \delta_{x,y,P} : \mathbf{1} + P &\to X \\ \mathrm{inl}(\star) &\mapsto x; \\ \mathrm{inr}(p) &\mapsto y; \end{aligned}$$

has a supremum $\bigvee \delta_{x,y,P}$ in X.

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- With excluded middle in $\mathcal{U}$, every poset is $\delta_{\mathcal{U}}$-complete.
- Assuming $x \neq y$, we have $\bigvee \delta_{x,y,P} = x \iff \neg P$, but $P \Rightarrow \bigvee \delta_{x,y,P} = y \Rightarrow \neg\neg P$.

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- With excluded middle in $\mathcal{U}$, every poset is $\delta_{\mathcal{U}}$-complete.
- Assuming $x \neq y$, we have $\bigvee \delta_{x,y,P} = x \iff \neg P$, but $P \Rightarrow \bigvee \delta_{x,y,P} = y \Rightarrow \neg\neg P$.
- If the two-element poset with $0 \sqsubseteq 1$ is $\delta_{\mathcal{U}}$-complete, then weak excluded middle holds in $\mathcal{U}$.

Examples of $\delta_{\mathcal{U}}$-complete posets

$\mathcal{U}$-sup-lattices (posets with all $\mathcal{U}$-suprema) are $\delta_{\mathcal{U}}$-complete, and so are $\mathcal{U}$-dcpos and $\mathcal{U}$-bounded complete posets.
(The family $\delta_{x,y,P}$ is bounded and directed when $x \sqsubseteq y$.)

Example

The $\mathcal{U}$-sup-lattices $\Omega_{\mathcal{U}}$ and $\mathcal{P}(X)$ for $X : \mathcal{U}$ are $\delta_{\mathcal{U}}$-complete.

Example

The $\mathcal{U}_0$-dcpos in the Scott model of PCF are $\delta_{\mathcal{U}_0}$-complete.

Making the theorem precise

Theorem (crude formulation)

Various kinds of nontrivial posets are small in HoTT/UF *if and only if propositional resizing holds.*

Items to be made precise

- ✓ Propositional resizing: $\Omega_{\mathcal{U}}$-Resizing and $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing.
- ✓ Various kinds of posets: $\delta_{\mathcal{U}}$-complete posets
- Nontrivial
- Small poset

Nontriviality and positivity

Definition

A poset is *nontrivial* if we have $x, y : X$ with $x \sqsubseteq y$ and $x \neq y$.

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- Nontriviality is very weak, because $x \neq y$ is a negated proposition.
- For $\delta_{\mathcal{U}}$-complete posets we can do better.

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- For $\delta_{\mathcal{U}}$-complete posets we can do better.

Definition

An element x of a $\delta_{\mathcal{U}}$-complete poset is *strictly below* an element y if

- $x \sqsubseteq y$ and
- for every $z \sqsupseteq y$ and proposition $P : \mathcal{U}$, we have $(z = \bigvee \delta_{x,z,P}) \Rightarrow P$.

Definition

A $\delta_{\mathcal{U}}$-complete poset X is *positive* if we have $x, y : X$ such that x is strictly below y.

Examples of nontriviality and positivity

Slogan

Positivity is to nontriviality what inhabitedness is to nonemptiness.

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Example

In the powerset, $\emptyset \neq A$ if and only if A is a nonempty subset, but $\emptyset$ is strictly below A if and only if A is an inhabited subset.

Example

In the type of propositions, $\mathbf{0} \neq P$ if and only if $\neg\neg P$ holds, but $\mathbf{0}$ is strictly below P if and only if P holds.

Making the theorem precise

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- ✓ Propositional resizing: $\Omega_{\mathcal{U}}$-Resizing and $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing.
- ✓ Various kinds of posets: $\delta_{\mathcal{U}}$-complete posets
- ✓ Positivity and nontriviality
- Small poset

(Locally) small $\delta_{\mathcal{U}}$-complete posets

Definition

A $\delta_{\mathcal{U}}$-complete poset $(X, \sqsubseteq)$ is *locally small* if the truth-value $x \sqsubseteq y$ is small for every $x, y : X$.

Example

Our running examples $\Omega_{\mathcal{U}}$ and $\mathcal{P}(X)$ for $X : \mathcal{U}$ are locally small, as are the large dcpos in the Scott model of PCF.

Definition

A $\delta_{\mathcal{U}}$-complete poset is *small* if it is locally small and its carrier is small.

Main results

Theorem

There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing holds.

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}$-Resizing holds.

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There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing holds.

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}$-Resizing holds.

Therefore, without resizing, there are no small nontrivial dcpos.

These are theorems of HoTT/UF. We do not make reference to models.

Proof sketch: using retracts

Definition

For a $\delta_{\mathcal{U}}$-complete poset X with points $x \sqsubseteq y$, we define

$$\begin{aligned} \Delta_{x,y} : \Omega_{\mathcal{U}} &\to X \\ P &\mapsto \bigvee \delta_{x,y,P} \end{aligned}$$

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Lemma

A locally small $\delta_{\mathcal{U}}$-complete poset X with points $x \sqsubseteq y$ is nontrivial if and only if the composite $\Omega_{\mathcal{U}}^{\neg\neg} \hookrightarrow \Omega_{\mathcal{U}} \xrightarrow{\Delta_{x,y}} X$ is a section.

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Lemma

A locally small $\delta_{\mathcal{U}}$-complete poset X with points $x \sqsubseteq y$ is positive if and only if for every $z \sqsupseteq y$, the map $\Omega_{\mathcal{U}} \xrightarrow{\Delta_{x,z}} X$ is a section.

Back to the main results

Lemma

If $s : A \to B$ is a section and B is a small set, then A is small too.

Theorem

There is a small nontrivial $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}^{\neg\neg}$-Resizing holds.

Theorem

There is a small positive $\delta_{\mathcal{U}}$-complete poset if and only if $\Omega_{\mathcal{U}}$-Resizing holds.

Decidable equality and excluded middle

Lemma

Types with decidable equality are closed under retracts.

Decidable equality and excluded middle

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Types with decidable equality are closed under retracts.

Constructively and predicatively, (locally small) $\delta_{\mathcal{U}}$-complete posets cannot have decidable equality and are necessarily large.

Theorem

There is a locally small nontrivial $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if weak excluded middle in $\mathcal{U}$ holds.

Theorem

There is a locally small positive $\delta_{\mathcal{U}}$-complete poset with decidable equality if and only if excluded middle in $\mathcal{U}$ holds.

Conclusion

Take-home message

- Nontrivial/positive sup-lattices, dcpos, bounded-complete posets, etc., can only be small if $\Omega^{\neg\neg}/\Omega$-resizing is assumed.
- Without propositional resizing, universe level management is necessary. In particular, the dcpos in the Scott model of PCF are necessarily large.
- Nontrivial/positive locally small sup-lattices, dcpos, etc., can only have decidable equality if (weak) excluded middle is assumed.

Conclusion

Take-home message

- Nontrivial/positive sup-lattices, dcpos, bounded-complete posets, etc., can only be small if $\Omega^{\neg\neg}/\Omega$-resizing is assumed.
- Without propositional resizing, universe level management is necessary. In particular, the dcpos in the Scott model of PCF are necessarily large.
- Nontrivial/positive locally small sup-lattices, dcpos, etc., can only have decidable equality if (weak) excluded middle is assumed.

Further results in our *FSCD'21* paper

- Various fixed point theorems crucially rely on propositional resizing.
- Zorn's Lemma implies propositional resizing (but not excluded middle).
- Compare completeness with respect to subsets/families.

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