

# Set-Theoretic and Type-Theoretic Ordinals Coincide

Tom de Jong<sup>1</sup>   Nicolai Kraus<sup>1</sup>  
Fredrik Nordvall Forsberg<sup>2</sup>   Chuangjie Xu<sup>3</sup>

<sup>1</sup>University of Nottingham, UK

<sup>2</sup>University of Strathclyde, UK

<sup>3</sup>SonarSource GmbH, Germany

Thirty-Eighth Annual ACM/IEEE Symposium on  
Logic in Computer Science (LICS)

29 June 2023

# Background and contribution

**Ordinals** are important in mathematical logic and computer science.

E.g., in the semantics of inductive data types, the justification of recursion and termination, the proof-theoretic strength of a formal system, etc.

# Background and contribution

**Ordinals** are important in mathematical logic and computer science.

E.g., in the semantics of inductive data types, the justification of recursion and termination, the proof-theoretic strength of a formal system, etc.

## Contributions

- 1 Working in **homotopy type theory (HoTT)**, we show that **set-theoretic** and **type-theoretic** approaches to ordinals coincide.

# Background and contribution

**Ordinals** are important in mathematical logic and computer science.

E.g., in the semantics of inductive data types, the justification of recursion and termination, the proof-theoretic strength of a formal system, etc.

## Contributions

- 1 Working in **homotopy type theory (HoTT)**, we show that **set-theoretic** and **type-theoretic** approaches to ordinals coincide.
- 2 We extend and generalize the above correspondence to *all* sets by considering certain **extensional wellfounded relations**.

This gives a new perspective on **Aczel's [1978]** type-theoretic interpretation of set theory.

# Ordinals in set theory

There are many classically equivalent notions of ordinals in set theory; the following is constructively acceptable [Powell 1975, Aczel–Rathjen 2010]:

**Def.** A set  $x$  is **transitive** if  $z \in y$  and  $y \in x$  implies  $z \in x$ .

**Def.** A **set-theoretic ordinal** is a transitive set whose elements are all transitive.

**Examples**  $0 := \emptyset$ ,  $1 := \{\emptyset\}$ ,  $2 := \{\emptyset, \{\emptyset\}\}$ ,  $\dots$ ,  $\mathbb{N} := \{0, 1, 2, \dots\}$ ,  $\dots$  are all set-theoretic ordinals.

# Ordinals in HoTT

In type theory, the statement “ $z : y$  and  $y : x$  implies  $z : x$ ” makes no sense. The [HoTT Book \[§10.3\]](#) instead defines ordinals as follows:

**Def.** A (type-theoretic) ordinal is a type  $X$  with a prop-valued binary relation  $<$  that is transitive, extensional and wellfounded.

**Example**  $(\mathbb{N}, <)$  is a type-theoretic ordinal.

# Ordinals in HoTT

In type theory, the statement “ $z : y$  and  $y : x$  implies  $z : x$ ” makes no sense. The [HoTT Book \[§10.3\]](#) instead defines ordinals as follows:

**Def.** A (type-theoretic) ordinal is a type  $X$  with a prop-valued binary relation  $<$  that is transitive, extensional and wellfounded.

**Example**  $(\mathbb{N}, <)$  is a type-theoretic ordinal.

Extensionality means that we have

$$x = y \iff \forall (u : X). (u < x \iff u < y).$$

# Ordinals in HoTT

In type theory, the statement “ $z : y$  and  $y : x$  implies  $z : x$ ” makes no sense. The [HoTT Book \[§10.3\]](#) instead defines ordinals as follows:

**Def.** A (type-theoretic) ordinal is a type  $X$  with a prop-valued binary relation  $<$  that is transitive, extensional and wellfounded.

**Example**  $(\mathbb{N}, <)$  is a type-theoretic ordinal.

Extensionality means that we have

$$x = y \iff \forall (u : X).(u < x \iff u < y).$$

Wellfoundedness is defined in terms of accessibility, but is equivalent to transfinite induction: for every  $P : X \rightarrow \mathcal{U}$ , we have  $\forall (x : X).P(x)$  as soon as  $\forall (x : X).(\forall (y : X).(y < x \rightarrow P(y))) \rightarrow P(x)$ .



# Ordinals in HoTT

In type theory, the statement “ $z : y$  and  $y : x$  implies  $z : x$ ” makes no sense. The [HoTT Book \[§10.3\]](#) instead defines ordinals as follows:

**Def.** A (type-theoretic) ordinal is a type  $X$  with a prop-valued binary relation  $<$  that is transitive, extensional and wellfounded.

**Example**  $(\mathbb{N}, <)$  is a type-theoretic ordinal.

**Def.** We write  $\mathbf{Ord}$  for the type of type-theoretic ordinals.

$\mathbf{Ord} ::= \Sigma(X : \mathcal{U}).\Sigma(< : X \rightarrow X \rightarrow \mathbf{Prop}).\text{“}< \text{ is transitive, ext. and wf.”}$

# The cumulative hierarchy in HoTT

We construct a type  $\mathbb{V}$  of **material sets**, known as the **cumulative hierarchy** [HoTT Book §10.5].

# The cumulative hierarchy in HoTT

We construct a type  $\mathbb{V}$  of **material sets**, known as the **cumulative hierarchy** [HoTT Book §10.5].

The type  $\mathbb{V}$  is a quotient inductive type with constructor

$$\mathbb{V}\text{-set} : (\Sigma(A : \mathcal{U}).(A \rightarrow \mathbb{V})) \rightarrow \mathbb{V}$$

# The cumulative hierarchy in HoTT

We construct a type  $\mathbb{V}$  of **material sets**, known as the **cumulative hierarchy** [HoTT Book §10.5].

The type  $\mathbb{V}$  is a quotient inductive type with constructor

$$\mathbb{V}\text{-set} : (\Sigma(A : \mathcal{U}).(A \rightarrow \mathbb{V})) \rightarrow \mathbb{V}$$

For example, the **empty set** is represented by  $\mathbb{V}\text{-set}(\mathbf{0}, \mathbf{0}\text{-rec})$ , and if  $x : \mathbb{V}$ , then the **singleton**  $\{x\}$  is represented by  $\mathbb{V}\text{-set}(\mathbf{1}, \lambda(u : \mathbf{1}).x)$ .

# The cumulative hierarchy in HoTT

We construct a type  $\mathbb{V}$  of **material sets**, known as the **cumulative hierarchy** [HoTT Book §10.5].

The type  $\mathbb{V}$  is a quotient inductive type with constructor

$$\mathbb{V}\text{-set} : (\Sigma(A : \mathcal{U}).(A \rightarrow \mathbb{V})) \rightarrow \mathbb{V}$$

quotiented by **bisimilarity**:  $\mathbb{V}\text{-set}(A, f)$  and  $\mathbb{V}\text{-set}(B, g)$  are identified exactly when  $f$  and  $g$  have the same image.

For example, the **empty set** is represented by  $\mathbb{V}\text{-set}(\mathbf{0}, \mathbf{0}\text{-rec})$ , and if  $x : \mathbb{V}$ , then the **singleton**  $\{x\}$  is represented by  $\mathbb{V}\text{-set}(\mathbf{1}, \lambda(u : \mathbf{1}).x)$ .

# The cumulative hierarchy in HoTT

We construct a type  $\mathbb{V}$  of **material sets**, known as the **cumulative hierarchy** [HoTT Book §10.5].

The type  $\mathbb{V}$  is a quotient inductive type with constructor

$$\mathbb{V}\text{-set} : (\Sigma(A : \mathcal{U}).(A \rightarrow \mathbb{V})) \rightarrow \mathbb{V}$$

quotiented by **bisimilarity**:  $\mathbb{V}\text{-set}(A, f)$  and  $\mathbb{V}\text{-set}(B, g)$  are identified exactly when  $f$  and  $g$  have the same image.

For example, the **empty set** is represented by  $\mathbb{V}\text{-set}(\mathbf{0}, \mathbf{0}\text{-rec})$ , and if  $x : \mathbb{V}$ , then the **singleton**  $\{x\}$  is represented by  $\mathbb{V}\text{-set}(\mathbf{1}, \lambda(u : \mathbf{1}).x)$ .

This is a refinement of **Aczel's [1978]** model of CZF in type theory (see also [Gylterud 2018]).

# Set-theoretic ordinals in HoTT

**Def.** We define **set membership**  $\in : \mathbb{V} \rightarrow \mathbb{V} \rightarrow \mathbf{Prop}$  by

$$x \in \mathbb{V}\text{-set}(A, f) \equiv \exists(a : A). f(a) = x.$$

# Set-theoretic ordinals in HoTT

**Def.** We define **set membership**  $\in : \mathbb{V} \rightarrow \mathbb{V} \rightarrow \text{Prop}$  by

$$x \in \mathbb{V}\text{-set}(A, f) := \exists(a : A). f(a) = x.$$

Using  $\in$ , we define the **subtype**  $\mathbb{V}_{\text{ord}}$  of  $\mathbb{V}$  of **set-theoretic ordinals** in HoTT:

$$\mathbb{V}_{\text{ord}} := \Sigma(x : \mathbb{V}). \text{“}x \text{ is a transitive set of transitive sets”}.$$



# Set-theoretic and type-theoretic ordinals coincide

Note:

- set membership  $\in$  is a wellorder on  $\mathbb{V}_{\text{ord}}$ ,
  - using initial segments, we can define a wellorder  $\prec$  on  $\text{Ord}$ ,
- so we have **type-theoretic ordinals**  $(\mathbb{V}_{\text{ord}}, \in)$  and  $(\text{Ord}, \prec)$ .

# Set-theoretic and type-theoretic ordinals coincide

Note:

- set membership  $\in$  is a wellorder on  $\mathbb{V}_{\text{ord}}$ ,
  - using initial segments, we can define a wellorder  $\prec$  on  $\text{Ord}$ ,
- so we have **type-theoretic ordinals**  $(\mathbb{V}_{\text{ord}}, \in)$  and  $(\text{Ord}, \prec)$ .

**Thm.** The type-theoretic ordinals  $(\mathbb{V}_{\text{ord}}, \in)$  and  $(\text{Ord}, \prec)$  are isomorphic and by univalence they are equal.

# Set-theoretic and type-theoretic ordinals coincide

Note:

- set membership  $\in$  is a wellorder on  $\mathbb{V}_{\text{ord}}$ ,
  - using initial segments, we can define a wellorder  $\prec$  on  $\text{Ord}$ ,
- so we have **type-theoretic ordinals**  $(\mathbb{V}_{\text{ord}}, \in)$  and  $(\text{Ord}, \prec)$ .

**Thm.** The type-theoretic ordinals  $(\mathbb{V}_{\text{ord}}, \in)$  and  $(\text{Ord}, \prec)$  are isomorphic and by univalence they are equal.

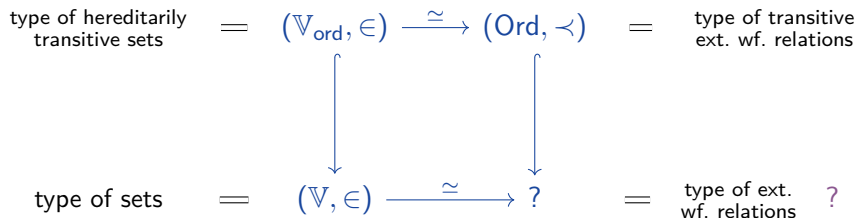
In HoTT,

*set-theoretic and type-theoretic ordinals coincide.*

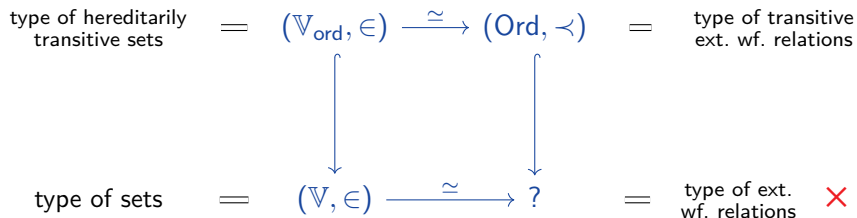
# Completing the square: from ordinals to sets

$$\begin{array}{l} \text{type of hereditarily} \\ \text{transitive sets} \end{array} = (\mathbb{V}_{\text{ord}}, \in) \xrightarrow{\cong} (\text{Ord}, \prec) = \text{type of transitive} \\ \text{ext. wf. relations} \\ \begin{array}{l} \text{type of sets} \\ \text{type of sets} \end{array} = (\mathbb{V}, \in) \xrightarrow{\cong} ?$$

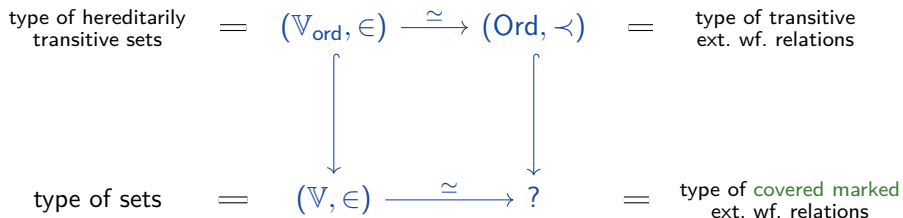
# Completing the square: from ordinals to sets



# Completing the square: from ordinals to sets



# Completing the square: from ordinals to sets



## Covered marked ext. wf. relations by example

We equip extensional wellfounded relations with a **marking** which picks out “top-level” elements.



## Covered marked ext. wf. relations by example

We equip extensional wellfounded relations with a **marking** which picks out “top-level” elements.

For example, the set  $\{\emptyset, \{\emptyset\}\}$  is represented by

$$\underline{0} < \underline{1},$$

while the set  $\{\{\emptyset\}\}$  is represented by

$$0 < \underline{1}.$$

## Covered marked ext. wf. relations by example

We equip extensional wellfounded relations with a **marking** which picks out “top-level” elements.

For example, the set  $\{\emptyset, \{\emptyset\}\}$  is represented by

$$\underline{0} < \underline{1},$$

while the set  $\{\{\emptyset\}\}$  is represented by

$$0 < \underline{1}.$$

A marking is **covering** if every element can be reached from a marked element, i.e., if the relation contains no “junk”.

# Summary

In HoTT, the **set-theoretic ordinals** in  $\mathbb{V}$  coincide with the **type-theoretic ordinals**.

By generalizing from type-theoretic ordinals to **covered marked ext. wf. relations**, we capture **all sets** in  $\mathbb{V}$ .

Question: Can we similarly capture **non-wellfounded** sets as certain graphs in HoTT?



**Full Agda formalisation.**

Building on Escardó's **TypeTopology**, and the **agda/cubical** library.

<https://tdejong.com/agda-html/st-tt-ordinals/>

# References

Peter Aczel. 'The type theoretic interpretation of constructive set theory'. In: *Logic Colloquium '77*. Ed. by A. MacIntyre, L. Pacholski and J. Paris. Vol. 96. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, 1978, pp. 55–66. DOI: [10.1016/S0049-237X\(08\)71989-X](https://doi.org/10.1016/S0049-237X(08)71989-X).

Peter Aczel and Michael Rathjen. 'Notes on Constructive Set Theory'. Book draft, available at: <https://www1.maths.leeds.ac.uk/~rathjen/book.pdf>. 2010.

Martín Hötzel Escardó et al. 'Ordinals in univalent type theory in Agda notation'. Agda development, HTML rendering available at: <https://www.cs.bham.ac.uk/~mhe/TypeTopology/Ordinals.index.html>. 2018.

Håkon Robbestad Gylterud. 'From Multisets to Sets in Homotopy Type Theory'. In: *The Journal of Symbolic Logic* 83.3 (2018), pp. 1132–1146. DOI: [10.1017/jsl.2017.84](https://doi.org/10.1017/jsl.2017.84).

William C. Powell. 'Extending Gödel's negative interpretation to ZF'. In: *The Journal of Symbolic Logic* 40.2 (1975), pp. 221–229. DOI: [10.1017/jsl.2017.84](https://doi.org/10.1017/jsl.2017.84).

The agda/cubical development team. *The agda/cubical library*. Available at: <https://github.com/agda/cubical/>. 2018–.

Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: <https://homotopytypetheory.org/book>, 2013.