Set-Theoretic and Type-Theoretic Ordinals Coincide

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Background and contribution

Ordinals are important in mathematical logic and computer science.

E.g., in the semantics of inductive data types, the justification of recursion and termination, the proof-theoretic strength of a formal system, etc.

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Contributions

1. Working in homotopy type theory (HoTT), we show that set-theoretic and type-theoretic approaches to ordinals coincide.
2. We extend and generalize the above correspondence to *all* sets by considering certain extensional wellfounded relations.

   This gives a new perspective on Aczel's [1978] type-theoretic interpretation of set theory.

Ordinals in set theory

There are many classically equivalent notions of ordinals in set theory; the following is constructively acceptable [Powell 1975, Aczel–Rathjen 2010]:

Def. A set x is transitive if $z \in y$ and $y \in x$ implies $z \in x$.

Def. A set-theoretic ordinal is a transitive set whose elements are all transitive.

Examples $0 \coloneqq \emptyset$, $1 \coloneqq \{\emptyset\}$, $2 \coloneqq \{\emptyset, \{\emptyset\}\}$, ..., $\mathbb{N} \coloneqq \{0, 1, 2, \ldots\}$, ... are all set-theoretic ordinals.

Ordinals in HoTT

In type theory, the statement "$z : y$ and $y : x$ implies $z : x$" makes no sense. The HoTT Book [§10.3] instead defines ordinals as follows:

Def. A (type-theoretic) ordinal is a type X with a prop-valued binary relation $<$ that is transitive, extensional and wellfounded.

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Wellfoundedness is defined in terms of accessibility, but is equivalent to transfinite induction: for every $P : X \to \mathcal{U}$, we have $\forall (x : X).P(x)$ as soon as $\forall (x : X).(\forall (y : X).(y < x \to P(y))) \to P(x)$.

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Def. We write Ord for the type of type-theoretic ordinals.

$$\mathsf{Ord} :\equiv \Sigma(X : \mathcal{U}).\Sigma(< : X \to X \to \mathsf{Prop}).\text{“}< \text{is transitive, ext. and wf.”}$$

The cumulative hierarchy in HoTT

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For example, the empty set is represented by $\mathbb{V}\text{-set}(\mathbf{0}, \mathbf{0}\text{-rec})$, and if $x : \mathbb{V}$, then the singleton $\{x\}$ is represented by $\mathbb{V}\text{-set}(\mathbf{1}, \lambda(u : \mathbf{1}).x)$.

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This is a refinement of Aczel's [1978] model of CZF in type theory (see also [Gylterud 2018]).

Set-theoretic ordinals in HoTT

Def. We define set membership $\in : \mathbb{V} \to \mathbb{V} \to \mathsf{Prop}$ by

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Using $\in$, we define the subtype $\mathbb{V}_{\mathsf{ord}}$ of $\mathbb{V}$ of set-theoretic ordinals in HoTT:

$$\mathbb{V}_{\mathsf{ord}} \;:\equiv\; \Sigma(x : \mathbb{V}).\text{“}x \text{ is a transitive set of transitive sets”}.$$

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Note:

- set membership $\in$ is a wellorder on $\mathbb{V}_{\mathsf{ord}}$,
- using initial segments, we can define a wellorder $\prec$ on Ord,

so we have type-theoretic ordinals $(\mathbb{V}_{\mathsf{ord}}, \in)$ and $(\mathsf{Ord}, \prec)$.

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In HoTT,

set-theoretic and type-theoretic ordinals coincide.

Completing the square: from ordinals to sets

$$\begin{array}{ccccccc} \text{type of hereditarily transitive sets} & = & (\mathbb{V}_{\mathsf{ord}}, \in) & \xrightarrow{\simeq} & (\mathsf{Ord}, \prec) & = & \text{type of transitive ext. wf. relations} \\ & & \big\downarrow & & \big\downarrow & & \\ \text{type of sets} & = & (\mathbb{V}, \in) & \xrightarrow{\simeq} & ? & & \end{array}$$

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type of hereditarily transitive sets	$=$	$(\mathbb{V}_{\mathsf{ord}}, \in)$	$\xrightarrow{\simeq}$	$(\mathsf{Ord}, \prec)$	$=$	type of transitive ext. wf. relations
		$\hookdownarrow$		$\hookdownarrow$		
type of sets	$=$	$(\mathbb{V}, \in)$	$\xrightarrow{\simeq}$	?	$=$	type of covered marked ext. wf. relations

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A marking is covering if every element can be reached from a marked element, i.e., if the relation contains no "junk".

Summary

In HoTT, the set-theoretic ordinals in $\mathbb{V}$ coincide with the type-theoretic ordinals.

By generalizing from type-theoretic ordinals to covered marked ext. wf. relations, we capture all sets in $\mathbb{V}$.

Question: Can we similarly capture non-wellfounded sets as certain graphs in HoTT?

Full Agda formalisation.
Building on Escardó's TypeTopology, and the agda/cubical library.
https://tdejong.com/agda-html/st-tt-ordinals/

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