Sharp Elements and the Scott Topology of Domains

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Introduction

Goal

Constructive study of the Scott topology of continuous dcpos.

Relevance of domain theory

Applications in:

- semantics of programming languages;
- higher-type computability;
- topology & locale theory.

Constructivity

- We don't assume excluded middle or the axiom of choice.
- Our results are valid in any topos (e.g. Hyland's effective topos).
- Constructive mathematics has been used for program extraction.
- Constructive domain theory may inspire computable domain theory.

Example of constructive benefits

- We can prove constructively that the Scott model of the programming language PCF is sound and computationally adequate.
- Given a PCF program t of base type and $n \in \mathbb{N}$,

$$[\![t]\!] = [\![\underline{n}]\!] \iff t \text{ reduces to } \underline{n}.$$

- Because the proof is constructive, we get an interpreter that runs a program, if we provide it with a proof that the program is *total*.

  The proof of totality can use topological/denotational ideas.

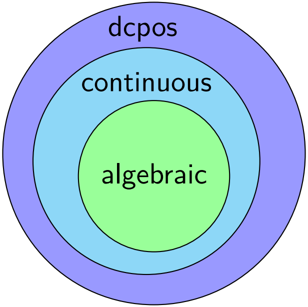
Outline

1. Preliminaries: basic domain theory
   - Examples of continuous dcpos
   - The Scott topology
2. The intrinsic apartness
3. Sharp elements
4. Strong maximality

Highlight and explain constructive aspects throughout the talk.

Basic objects in domain theory

- In domain theory we study directed complete posets (dcpos): posets with suprema for directed subsets.
- All our examples will be continuous dcpos which have a basis whose elements can be used to *approximate* any other element.
- In the special case of algebraic dcpos every element can be *approximated* using compact ("finitary") elements.



Examples of continuous dcpos I

Example

- The powerset $\mathcal{P}(X)$ of any set X, ordered by subset inclusion, is an algebraic dcpo.
- Its compact elements are exactly the *Kuratowski finite* subsets.
  A subset S is Kuratowski finite if it is finitely enumerable, possibly with repetitions.
- Since $\mathbb{N}$ has decidable equality, we can get rid of repetitions in Kuratowski finite subsets of $\mathbb{N}$.
- But we cannot do this for subsets of real numbers, because $\mathbb{R}$ does not have decidable equality.

Examples of continuous dcpos II

Another class of examples is given by considering rounded ideals on abstract bases.

Example

- Consider the poset $(A^*, \preceq)$ of finite sequences on some set A ordered by prefix.
- We get an algebraic dcpo $\mathcal{A}$ whose compact elements are precisely of the form ${\downarrow}\tau := \{\sigma \in A^* \mid \sigma \preceq \tau\}$ with τ a finite sequence.
- We have an injection from the set of infinite sequences to $\mathcal{A}$ given by $\alpha \mapsto \{\sigma \in A^* \mid \sigma \text{ is a prefix of } \alpha\}$.

Example

Taking rounded ideals on $(\mathbb{Q}, <)$ we get a continuous dpco of lower Dedekind reals.

Examples of continuous dcpos III

Example

- Consider the set $\mathbb{Q} \times_{<} \mathbb{Q} :\equiv \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}$ ordered by $(p,q) \prec (r,s) \iff p < r < s < q$.
- Taking rounded ideals we get a continuous dcpo $\mathcal{R}$ of partial (two-sided) Dedekind reals that classically, is isomorphic to the well-known interval domain.
- The partial reals are bounded, rounded and transitive, but may fail to be located: a real x is *located* if for every $p, q \in \mathbb{Q}$ with $p < q$ we have $p < x$ or $x < q$.
- We may think of a partial real as a *computation* of rational approximations which may get *stuck*.
- We have an injection from $\mathbb{R}$ to $\mathcal{R}$ given by $x \mapsto \{(p,q) \mid p < x \text{ and } x < q\}$.

The Scott topology

Definition

A subset C of a dcpo D is *Scott closed* if

- it is a *lower set*: if $x \sqsubseteq y \in C$, then $x \in C$;
- it is closed under directed suprema: $\bigsqcup S \in C$ for directed $S \subseteq C$.

Definition

A subset U of a dcpo D is *Scott open* if

- it is an *upper set*.
- it is inaccessible by directed suprema: if $\bigsqcup S \in U$, then $s \in U$ for some $s \in S$ for every directed $S \subseteq D$.

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- Complements of Scott opens are Scott closed.
- But, complements of Scott closed subsets are Scott open if and only if excluded middle holds.

Introduction to apartness

- For real numbers $x \neq y$ is very weak, because it is a *negative* notion.
- Brouwer introduced apartness on real numbers, a *positive* notion:

$$x \mathrel{\#} y \iff \exists_{q \in \mathbb{Q}} (x < q < y) \vee (y < q < x).$$

- Classically, $x \mathrel{\#} y \iff x \neq y$, but constructively, $x \mathrel{\#} y$ is stronger.

- Apartness on real numbers is used by Brouwer and in Bishop's *Foundations of Constructive Analysis* (1976).
- Bridges and Vîță developed general topology using apartness relations in *Apartness and Uniformity* (2011).

Intrinsic apartness

Classically, every dcpo with the Scott topology is T_0-separated:

1. if x and y have the same Scott open neighbourhoods, then $x = y$;
2. if $x \neq y$, then there is a Scott open neighbourhood of x that does not contain y or vice versa.

- Item 1 holds constructively for continuous dcpos.
- Item 2 is equivalent to excluded middle.

Intrinsic apartness

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This leads to our first new notion.

Definition

Two elements $x, y \in D$ are *intrinsically apart*, written $x \# y$, if there is a Scott open that contains one but not the other.

Intrinsic apartness in examples

Example

For two subsets $A, B \in \mathcal{P}(X)$ we have $A \# B$ if and only if $(B \setminus A)$ or $(A \setminus B)$ is inhabited.

Example

- If α and β are two infinite sequences on a set A, then we may say that α and β are apart if $\exists_{n \in \mathbb{N}} (\alpha(n) \neq \beta(n))$.
- Recall the injection $\iota \colon \alpha \mapsto \{\sigma \in A^* \mid \sigma \text{ is a prefix of } \alpha\}$ into the continuous dcpo $\mathcal{A}$.
- Then, α and β are apart as sequences if and only if $\iota(\alpha)$ and $\iota(\beta)$ are intrinsically apart.

Example

Similarly, two real numbers are apart in Brouwer's sense if and only if they are intrinsically apart in our sense as partial Dedekind reals.

Cotransitivity and tightness I

Definition

A relation $\sharp$ is:

- *cotransitive* if $x \# y$ implies $x \# z$ or $z \# y$ for every x, y and z.
- *tight* if $\neg(x \sharp y)$ implies $x = y$.

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Warning

We only ask that an apartness is irreflexive and symmetric, we do not require cotransitivity or tightness.

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Proposition

The intrinsic apartness is cotransitive/tight if and only if excluded middle holds.

But we will get cotransitivity and tightness in special cases discussed later.

Cotransitivity and tightness II

Is the intrinsic apartness at fault for not being cotransitive or tight?
No, because of the following result.

Theorem

If $\sharp$ is any irreflexive relation on a dcpo with elements $x \sqsubseteq y$ such that $x \sharp y$, then cotransitivity of $\sharp$ implies weak excluded middle.

Sharp elements

- For each dcpo, we can define a subset of sharp elements.
- Sharp elements occur naturally in the examples.
- The intrinsic apartness *is* tight and cotransitive on sharp elements.

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Proposition

An element x of an algebraic dcpo is sharp if and only if $c \sqsubseteq x$ is decidable for every compact element c.

Proposition

Every element is sharp if and only if excluded middle holds.

Examples of sharp elements

Example

The sharp elements of $\mathcal{P}(X)$ are exactly the decidable subsets of X.

Example

- We can try to make a one-sided lower real into a two-sided real: given $L \subseteq \mathbb{Q}$, define $U :\equiv \{q \in \mathbb{Q} \mid \exists_{p \in \mathbb{Q}} (p \notin L \wedge p < q)\}$.
- (L, U) is bounded, directed and transitive.
- Classically, (L, U) is located too, but:
- (L, U) is located if and only if L is sharp in the dcpo of lower reals.

Slogan

Sharpness generalizes locatedness.

Sharpness is also related to a notion of locatedness in formal topology due to Spitters and Kawai.

Cotransitivity, tightness and sharpness

Theorem

In a continuous dcpo, tightness and cotransitivity hold for sharp elements:

- *If x and y are sharp, then $\neg(x \# y)$ implies $x = y$.*
- *If z is sharp and x and y are arbitrary, then $x \# y$ implies $x \# z$ or $z \# y$.*

Strong maximality, classically

Subspaces of maximal elements

Many topological spaces are homeomorphic to the maximal elements of some continuous dcpo with the relative Scott topology.

Example

The real line is homeomorphic to the subspace of maximal partial Dedekind reals.

Strong maximality, classically

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Strong maximality

- In general, the subspace of maximal elements may fail to be Hausdorff.
- Therefore, Smyth and Heckmann studied strong maximality: the subspace of strongly maximal elements is both Hausdorff and regular.
- Strong maximality and maximality coincide if and only if the Lawson condition holds.

Strong maximality, constructively

Proposition

Every strongly maximal element is sharp.

Example

- The real line is homeomorphic to the subspace of strongly maximal partial Dedekind reals.
- If every maximal partial Dedekind real is strongly maximal, then weak excluded middle follows.

Can we expect (strong maximality $=$ maximality $+$ sharpness)?

Examples: strong maximality, maximality & sharpness

Example

- We have an algebraic dcpo $\mathcal{C}$ built from binary sequences.
- Cantor space $\mathbf{2}^{\mathbb{N}}$ is homeomorphic to the subspace of *strongly maximal* elements of $\mathcal{C}$.
- For $\mathcal{C}$ we have: strong maximality $=$ maximality $+$ sharpness.

Example

- We have an algebraic dcpo $\mathcal{B}$ built from sequences of natural numbers.
- Baire space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the subspace of *strongly maximal* elements of $\mathcal{B}$.
- For $\mathcal{B}$ we have: (strong maximality $=$ maximality $+$ sharpness) $\iff$ Markov's Principle (MP).
- MP: Every binary sequence that is not all zeroes must have a one.

Summary

Studied Scott topology on dcpos constructively featuring many examples.

- Intrinsic apartness is a *positive* counterpart to the negation of equality.
- The intrinsic apartness is very well-behaved w.r.t. sharp elements.
- Strongly maximal elements are sharp and the notion is very useful constructively, often more so than maximality.

Further result in the paper

The Bridges-Vîţă framework applies to domain theory: in many continuous dcpos, the Scott topology coincides with the apartness topology induced by the intrinsic apartness.

Extended version of the MFPS paper

Sharp Elements and the Scott Topology of Continuous Dcpos. arXiv: 2106.05064 (math.LO).