Introduction to homotopy theory type / univalent foundations

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10th autumn school *Proof and Computation*

14 September 2026, Fischbachau, Germany — Lecture 1/3

In short

Homotopy type theory / univalent foundations (HoTT/UF) is

- a foundation for mathematics,
- a suitable language for higher mathematical structures,
- implemented in computer proof assistants.

As a foundation, it is unlike set theory. Some notable differences:

- The syntax of set theory is both very minimal and unrestrictive.
  The syntax of HoTT is neither.
- In set theory, logic comes first and equality is a logical concept.
  In HoTT, logic comes second (*after* mathematics) and
  equality (identification) is a mathematical concept (a type).

Course overview

To be adjusted as needed

Lecture 1: Introduction, historical overview, discussion of resources

Lecture 2: Three topics:

- constructivism & computation
- key concepts from univalent foundations
- syntax of Martin-Löf Type Theory

Lecture 3: Logic in HoTT/UF and the differences with propositions in other type theories

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It's beneficial and **fun** to explore HoTT/UF in a proof assistant, such as Agda.

Come and talk to me if you'd like to learn Agda; we can set up a working group!

An abridged and incomplete history

Timeline

1970s

Per Martin-Löf presents his intensional dependent type theories (with publications in the 1980s). They feature **identity types**.

Later: **MLTT** = Martin-Löf Type Theory.

1994

Martin Hofmann and **Thomas Streicher** publish their groupoid model of MLTT.

This is a "1-dimensional homotopical" model and shows that Martin-Löf's identity types can have **higher structure**.

1998

Hofmann and Streicher publish an extended version of their 1994 paper featuring "universe extensionality": a precursor to Voevodsky's univalence axiom.

2006

Workshop *Identity Types: Topological and Categorical Structure* in Uppsala, Sweden, feat. converging talks by Thomas Streicher, **Michael Warren**, **Steve Awodey**, **Richard Garner** and **Benno van den Berg**.

2006

Field Medallist **Vladimir Voevodsky** makes a note on "a homotopy λ-calculus" public.

2009

Voevodsky starts working out an interpretation of MLTT in *simplicial sets* (think: "spaces").

2010

Voevodsky starts his formal Coq library *Foundations*, which is later merged to become part of *(Coq-)UniMath*.

2011

Awodey, Garner, Martin-Löf and Voevodsky organize the Oberwolfach mini-workshop *The Homotopy Interpretation of Constructive Type Theory*.

Andrej Bauer, Peter Lumsdaine, Mike Shulman and Michael Warren conceive of **higher inductive types**.

2012–2013

A Special Year on Univalent Foundations of Mathematics is held at the Institute for Advanced Study, culminating in the collaboratively written **HoTT Book**.

2014, 2015–2021

Marc Bezem, **Thierry Coquand** and Simon Huber give a **constructive** model of HoTT in *cubical* (as opposed to *simplicial*) sets.

This gives rise to a computational version of HoTT: **cubical type theory** by Cyril Cohen, Anders Mörtberg, Coquand and Huber, and later to *Cubical Agda* by Andrea Vezossi, Andreas Abel and Mörtberg.

2019

Mike Shulman's preprint vastly generalizes Voevodsky's simplicial sets model and shows HoTT can be interpreted into any $(\infty, 1)$-topos.

Recent research line: extensions of HoTT to do **synthetic mathematics**:

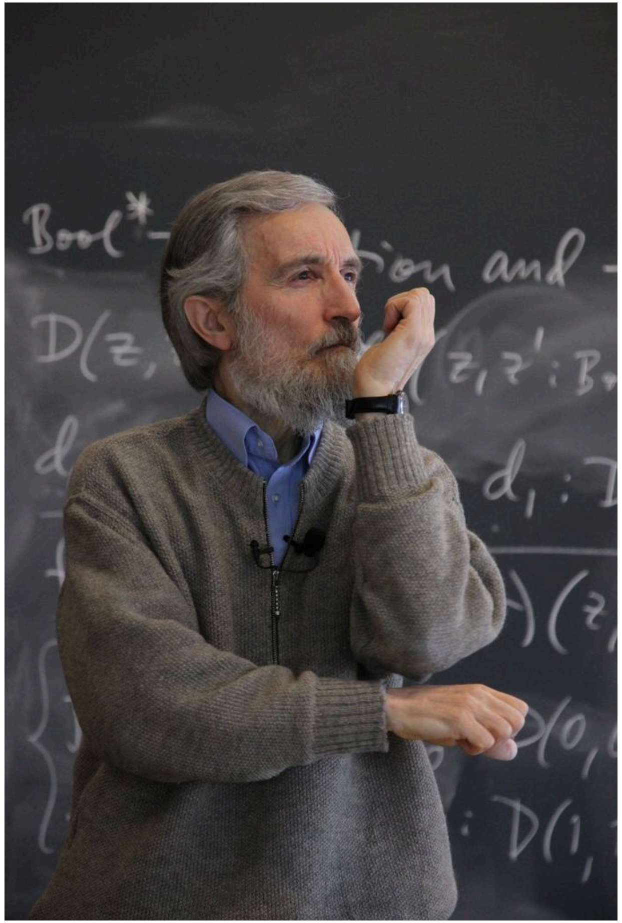
Idea: use a "domain-specific language" with fewer encodings ↲

- synthetic $(\infty, 1)$-category theory
  ↳ Riehl–Shulman, Buchholtz–Gratzer–Weinberger, Sattler–Wärn, …
- synthetic algebraic geometry
  ↳ Blechschmidt, Cherubini, Coquand, Ritter (prev. Hutzler), Moeneclaey, …

Identity types, groupoids and homotopy theory

Martin-Löf's identity types

- For every type A and terms $a, b : A$, there is a new type $\mathrm{Id}_A(a, b)$ of identities/identifications between a and b.
  - This is Martin-Löf's *formation* rule for the identity type.



Per Martin-Löf

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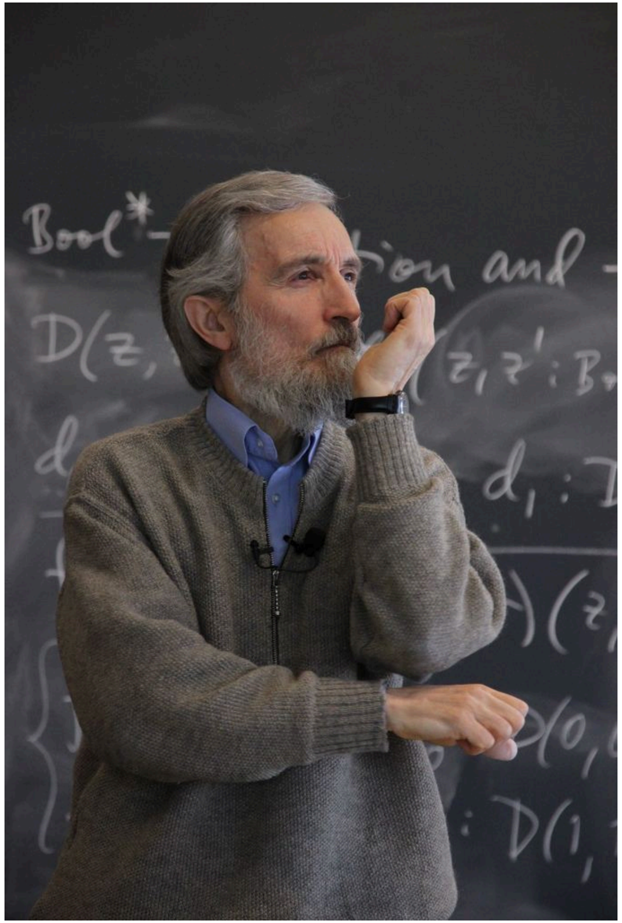
- **Reflexivity**: For every $a : A$, we have a term $\mathrm{refl}\, a : \mathrm{Id}_A(a, a)$.
  - This is Martin-Löf's *introduction rule* for the identity type.



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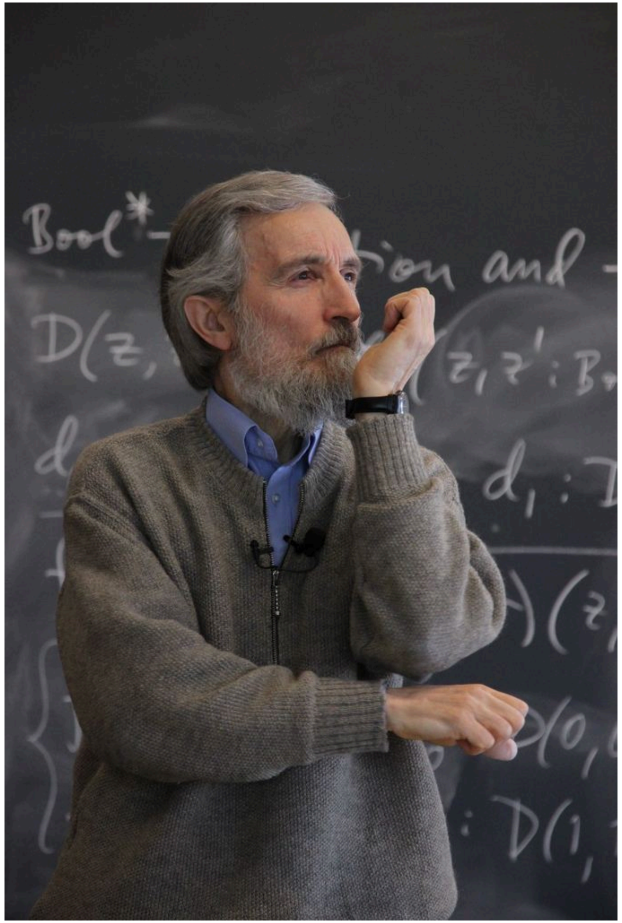
- **Reflexivity**: For every $a : A$, we have a term $\mathrm{refl}\, a : \mathrm{Id}_A(a, a)$.
  - This is Martin-Löf's *introduction rule* for the identity type.
- We can construct **symmetry** and **transitivity** witnesses, e.g. if we have $p : \mathrm{Id}_A(a, b)$ and $q : \mathrm{Id}_A(b, c)$, then we can construct a new element $r : \mathrm{Id}_A(a, c)$.
  - This is done using Martin-Löf's *elimination rule* for the identity type. It's also known as the **J-rule**.



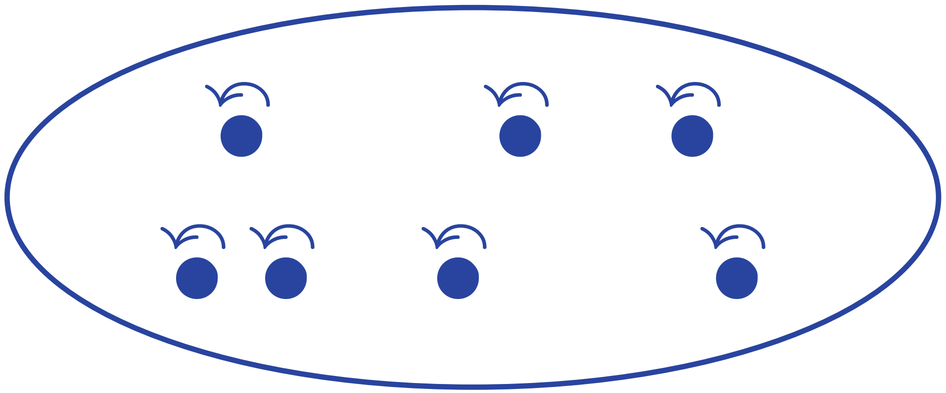
Per Martin-Löf

Thomas Streicher (1958–2025)

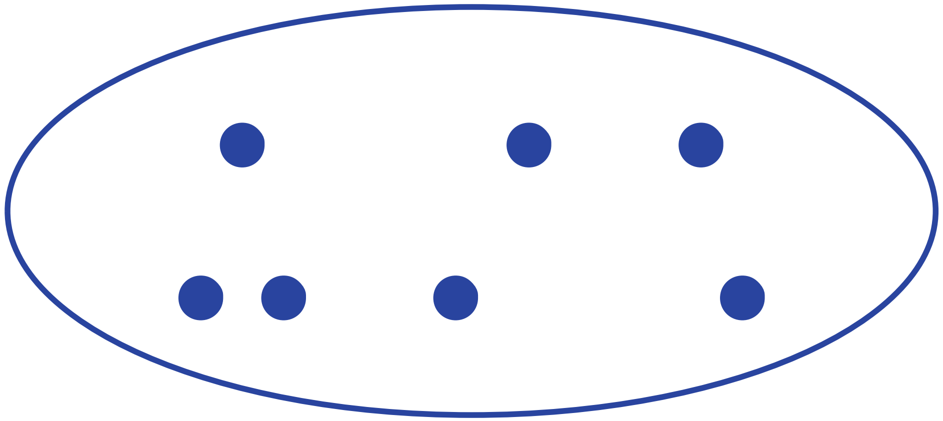
Maybe you'd expect that every $p : \mathrm{Id}_A(a, a)$ must be $\mathrm{refl}\, a$?

- This is *not* provable in MLTT!
- In HoTT there will be types A with points $a : A$, for which this is false, and this will be a **feature**, not a bug!
- But in ordinary MLTT, you can consistently add it as an axiom. In fact, it's known as **Streicher's Axiom K**.

Under Axiom K, types just look like a collection of points:



An illustration of a type with some elements depicted as • with depicting refl at that point.



It is common to omit the reflexivities, so that we draw the above just as on the left.

Hofmann & Streicher's groupoid model

- **Definition.** A **groupoid** is a set X together with for every $x, y \in X$, a set of "paths" from x to y such that
  - we always have a path $\mathrm{id}_x : x \rightsquigarrow x$,
  - we can compose paths: given $p : x \rightsquigarrow y$ and $q : y \rightsquigarrow z$ we have $q \circ p : x \rightsquigarrow z$, and this operation must be associative,
  - we can invert a path $p : x \rightsquigarrow y$ to $p^{-1} : y \rightsquigarrow x$ such that $p^{-1} \circ p = \mathrm{id}_x$ and $p \circ p^{-1} = \mathrm{id}_y$.

Martin Hofmann
(1965–2018)

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- Trivial example: set X where the *only* paths are $\mathrm{id}_x : x \rightsquigarrow x$.
- Nontrivial example: $X = \{*\}$, a path $* \rightsquigarrow *$ is an integer $k \in \mathbb{Z}$, $\mathrm{id}_* = 0$, composition is addition of integers, and k^{-1} is $-k$.

Martin Hofmann
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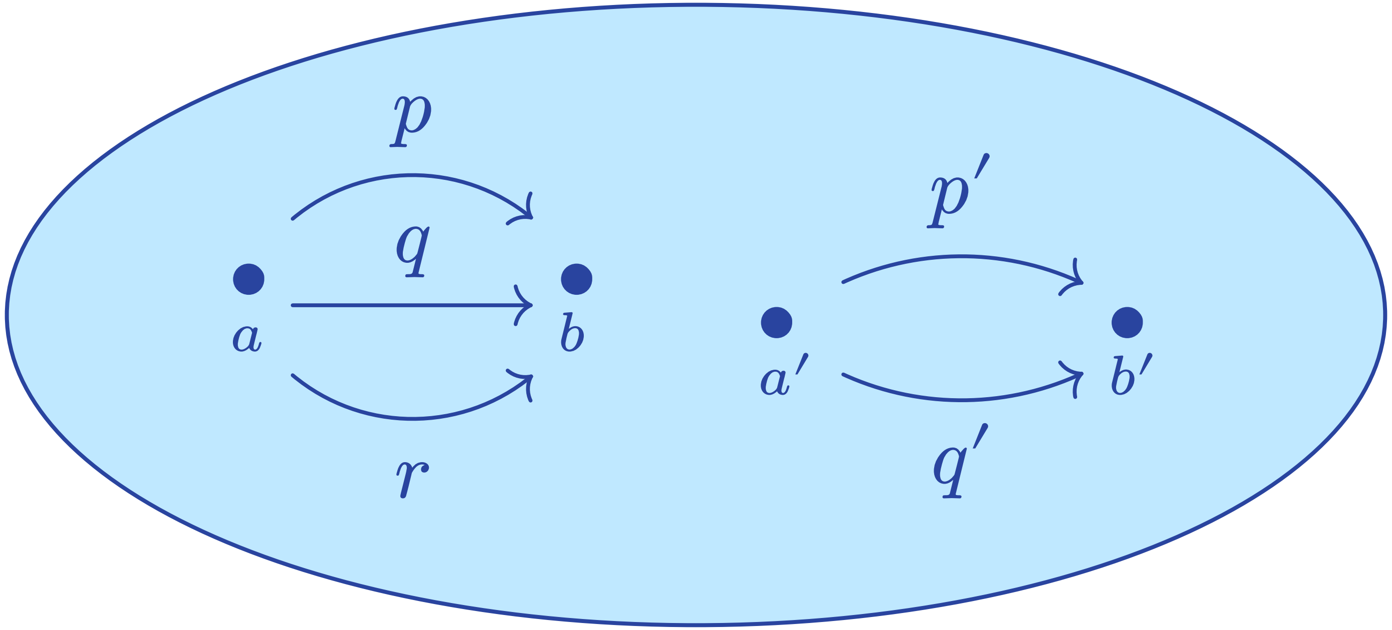
- Nontrivial example: $X = \{*\}$, a path $* \rightsquigarrow *$ is an integer $k \in \mathbb{Z}$, $\mathrm{id}_* = 0$, composition is addition of integers, and k^{-1} is $-k$.
- In the model, a type A is a groupoid $[\![A]\!]$ and $p : \mathrm{Id}_{A(a,b)}$ is a path $[\![a]\!] \rightsquigarrow [\![b]\!]$ in $[\![A]\!]$.

Martin Hofmann
(1965–2018)

In the groupoid model, types look like a collection of points **with paths** between them:



The Uppsala workshop — Beyond groupoids

Identity Types - Topological and Categorical Structure Workshop

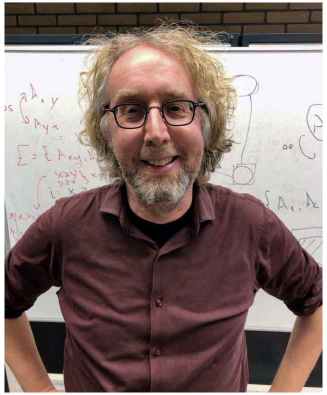
https://www2.math.uu.se/~palmgren/itt/

Theme of workshop *The identity type is one of the most intriguing constructions of Martin-Löf type theory. Its complexity became apparent with the Hofmann–Streicher groupoid model of type theory.*

This model also hinted at some possible connections between type theory and homotopy theory and higher categories. Exploration of this connection is intended to be the main theme of the workshop.

Michael Warren

Steve Awodey



Benno van den Berg

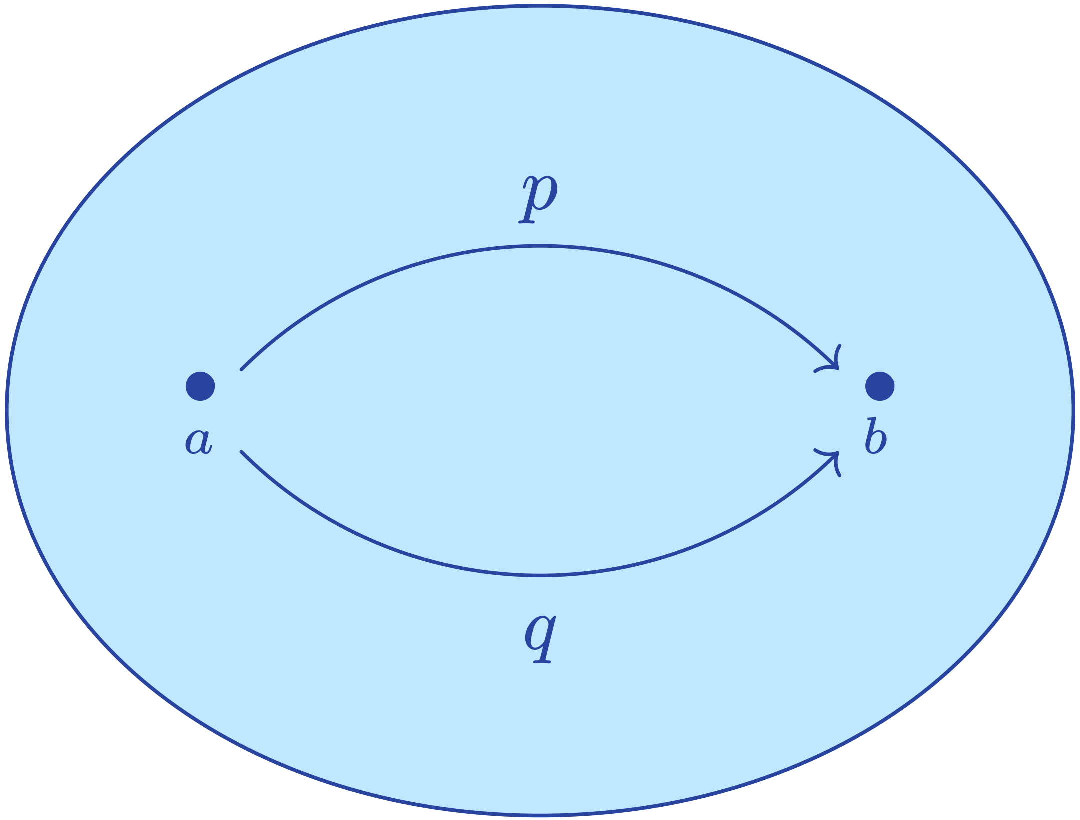
Richard Garner

In HoTT, we picture types as **higher groupoids** that in addition to paths between points, can have (higher) paths between paths.

A Type

$a, b : A$

$p, q : \mathrm{Id}_A(a, b)$



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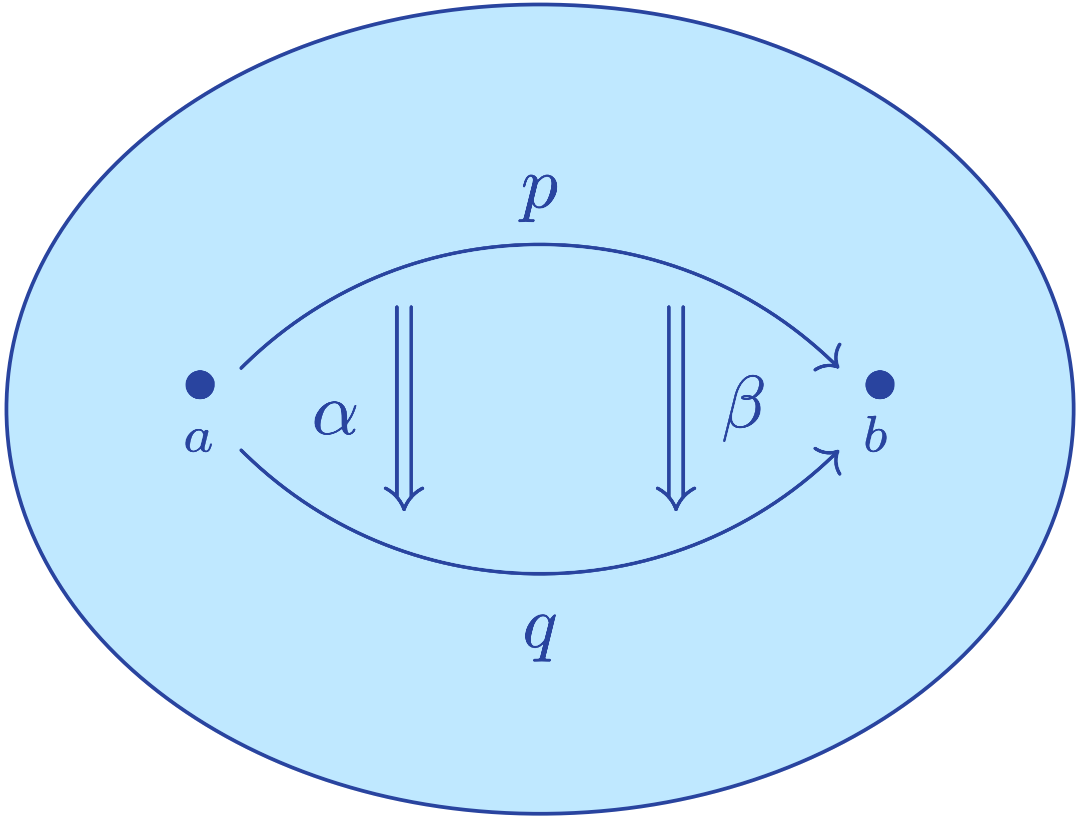
$$A \ \ \mathrm{Type}$$

$$a, b : A$$

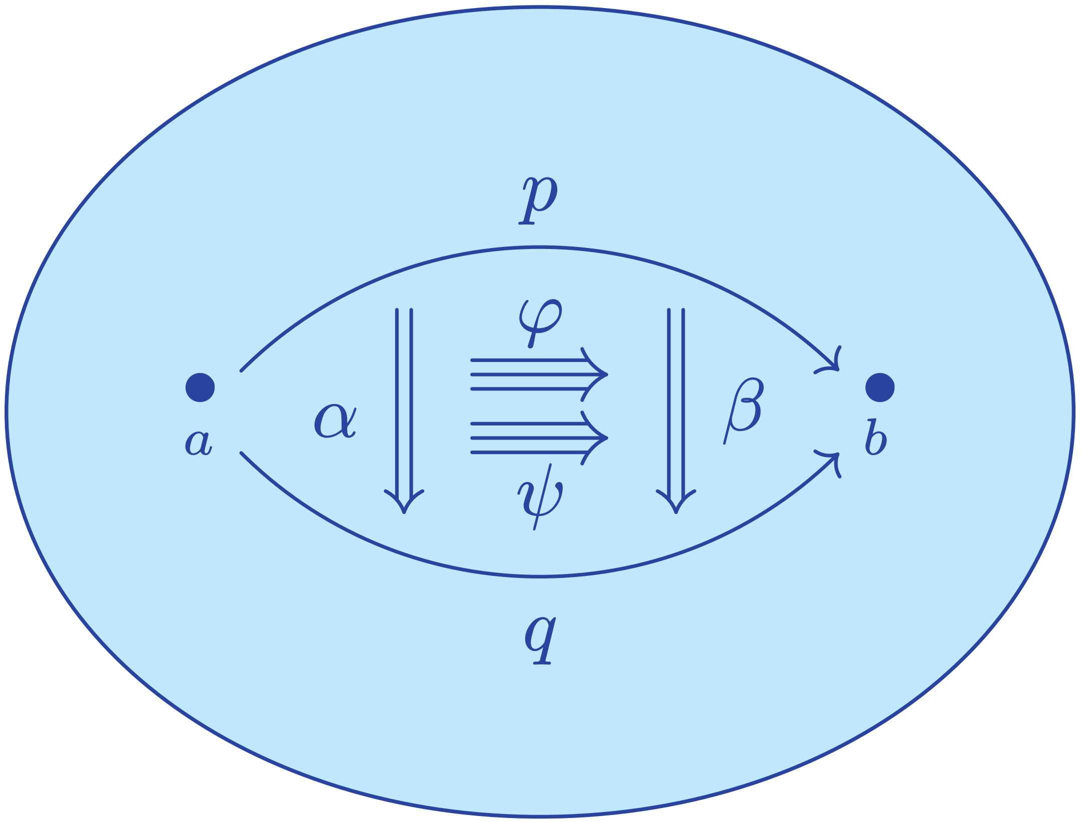
$$p, q : \mathrm{Id}_A(a, b)$$

$$\alpha, \beta : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p, q)$$



In HoTT, we picture types as **higher groupoids** that in addition to paths between points, can have (higher) paths between paths.

$$A \ \text{Type}$$
$$a, b : A$$
$$p, q : \mathrm{Id}_A(a, b)$$
$$\alpha, \beta : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p, q)$$
$$\varphi, \psi : \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}(\alpha, \beta)$$
$$\vdots$$



Voevodsky's univalent foundations

Voevodsky's journey into foundations

Some quotes from a lecture by Voevodsky
https://www.ias.edu/ideas/2014/voevodsky-origins

- *Starting from 1993 multiple groups of mathematicians studied [my] "Cohomological Theory" paper at seminars and used it in their work and none of them noticed the mistake.*

  And it clearly was not an accident. ***A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail.***

Vladimir Voevodsky
(1966–2017)

- [On the mathematics of his “2-theories”]

  To do the work at the level of rigor and precision I felt was necessary would take an enormous amount of effort and would produce a text that would be very difficult to read. And ***who would ensure that I did not forget something and did not make a mistake****, if even the mistakes in much more simple arguments take years to uncover?*

  I think it was at this moment that I largely stopped doing what is called “curiosity driven research” and started to think seriously about the future.
  It soon became clear that the only real long-term solution to the problems [...] is to ***start using computers in the verification of mathematical reasoning****.*

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  I think it was at this moment that I largely stopped doing what is called “curiosity driven research” and started to think seriously about the future.
  *It soon became clear that the only real long-term solution to the problems [...] is to **start using computers in the verification of mathematical reasoning**.*

😉 Some words [emphasis mine] of encouragement for you *and* me:

*I would like to thank all of those who are trying to understand the ideas of Univalent Foundations, to develop these ideas and to communicate these ideas to others. **I know it is difficult**.*

[TYPES/announce] homotopy lambda calculus

Vladimir Voevodsky vladimir at ias.edu
Fri Sep 29 10:06:07 EDT 2006

- Previous message: [TYPES/announce] INRIA postdoc position
- Next message: [TYPES/announce] Full Professor FormalMethods and Tools Twente
- **Messages sorted by:** [date] [thread] [subject] [author]

```
I have recently produced a short note where I try to explain what
homotopy lambda calculus is (or is supposed to be). Here it is:



-------------- next part --------------
A non-text attachment was scrubbed...
Name: 2006_09_Hlambda.pdf
Type: application/pdf
Size: 116584 bytes
Desc: not available
Url : http://lists.seas.upenn.edu/pipermail/types-announce/attachments/20060929/c8464d57/2006_09_Hlambda.pdf
-------------- next part --------------


Vladimir (Voevodsky)
```

Voevodsky's email to the TYPES/announce mailing list on 29 September 2006

https://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations_files/Hlambda_short_current.pdf

Voevodsky's univalent foundations

The simplicial sets model of HoTT is very relevant and important, but at least equally so are the novelties of Voevodsky's *Foundations* Coq library.

The significance lied in his **new and elegant definitions**, among them:

- n-(truncated) types (called "hlevels" by Voevodsky), with **contractible** types as a fundamental instance;
- the **fiber** of a function;
- a well behaved notion of **equivalence**;
- the **univalence axiom** which connects equivalences with Martin-Löf's identity types.

More on this in the coming lectures.

Writing mathematics in type theory

Informal type theory

Peter Aczel (1941–2023)

Mathematicians relying on set theory don't work with the *formal* rules of set theory.

One innovation of the special year, spurred by Aczel, was to develop mathematics in type theory, in an *informal* (but still rigorous!) way.

To see the effect of "informal type theory", **compare** the following excerpt from the 2017 paper *Notions of Anonymous Existence in Martin-Löf type Theory* by Kraus, Escardó, Coquand and Altenkirch

Lemma 3.2. *If a type has decidable equality, its path spaces have constant endofunctions:*

$$\mathsf{isDiscrete}\,X \to \mathsf{pathConstEndo}\,X. \tag{3.2}$$

Proof. Given inhabitants x and y of X, we get by assumption either an inhabitant of $x = y$ or an inhabitant of $\neg(x = y)$. In the first case, we construct the required constant function $(x = y) \to (x = y)$ by mapping everything to this given path. In the second case, we have a proof of $\neg(x = y)$, and the canonical function is constant automatically. $\square$

with the original treatment in Hedberg's 1998 *A coherence theorem for Martin-Löf's type theory* on the next slide.

On any decidable set a constant function may be defined with values in the same set. The proof will give two families of functions (for an arbitrary set A). Recall that $\mathscr{D}(A)$ stands for (the set representation of) the formula 'A or not A'.

$$\begin{array}{ll} con_A(z) : A \to A & [z \in \mathscr{D}(A)] \\ iscon_A(z,x,y) \in I(con_A(z,x), con_A(z,y)) & [z \in \mathscr{D}(A); x, y \in A] \end{array}$$

Assume we have a set A. By $+$-elimination, applied to the constant family of sets $A\ [z \in \mathscr{D}(A)]$, we define a function $con_A : \mathscr{D}(A) \to A \to A$ subject to the definitional equalities

$$\begin{cases} con_A(i(x), a) &= x \quad [x \in A] \\ con_A(j(f), a) &= a \quad [f \in (\Pi x \in A) N_0] \end{cases}$$

and, to define the function $iscon_A$, i.e. to prove that $con_A(d)$ is a constant function for an arbitrary element d of $\mathscr{D}(A)$, we assume that we have two elements a and a' of A, and consider the family of propositions (sets) C given by $C(z) = con_A(z,a) \simeq con_A(z,a')\ [z \in \mathscr{D}(A)]$.

By $+$-elimination, $C(z)$ is proved for arbitrary $z \in \mathscr{D}(A)$ when we have proved the special cases, $C(i(x))[x \in A]$ and $C(j(f))[f \in (\Pi x \in A) N_0]$. Recalling the definition of C and the equations of con, this leaves us with proving, after simplification, in the first case, that $I(x,x)$ is inhabited for $x \in A$, and in the second, that $I(a,a')$ is inhabited for $f \in (\Pi x \in A) N_0$. The first case is solved by reflexivity of identity, and the second case is solved by absurdity elimination – we can apply f to any of the two elements a and a' of A, to get an element of N_0.

```
con ∈ (A ∈ Set;
       d ∈ Dec(A);
       a ∈ A
      )A
  con ≡
      [A, d, a]
          when(A,
               Prod(A, [h]N0),
               [h]A,
               [x]x,
               [f]a,
               d)
iscon ∈ (A ∈ Set;
         d ∈ Dec(A);
         a, a' ∈ A
        )I(A, con(A, d, a), con(A, d, a'))
  iscon ≡
      [A, d, a, a']
          when(A,
               Prod(A, [h]N0),
               [z]I(A, con(A, z, a), con(A, z, a')),
               [x]ref(A, x),
               [f]case0([h]I(A, a, a'), app(A, [h]N0, f, a)),
               d)
```

Since the HoTT Book

Earlier work on HoTT often featured fairly elaborate calculations, usually involving transport. Since then, we have found abstractions to handle this more swiftly, e.g.:

- **equational reasoning with equivalences**, cf. my expository paper Formalizing equivalences without tears.
- **characterizing identity types via general machinery**
  - structure identity principles (Coquand & Danielsson; Escardó; Ahrens, North, Shulman, Tsementzis; Angiuli, Cavallo, Mörtberg, Zeuner).
  - fundamental theorem of identity types (Rijke; Escardó (“Yoneda”)) vs *encode-decode*
- descent/flattening to deal with pushouts

Resources

My slides and notes

- The slides for this lecture series are available at https://tomdjong.com/teaching.html.
- In August, I taught a **5×90 min** introductory course on HoTT/UF at the European Summer School in Logic, Language and Information (ESSLLI).

  You can find the material for that bigger course at https://github.com/tomdjong/ESSLLI-2026
- If you want to learn more, you can
  1. read the ESSLLI notes,
  2. consult the resources in the following slides,
  3. **talk to me!**

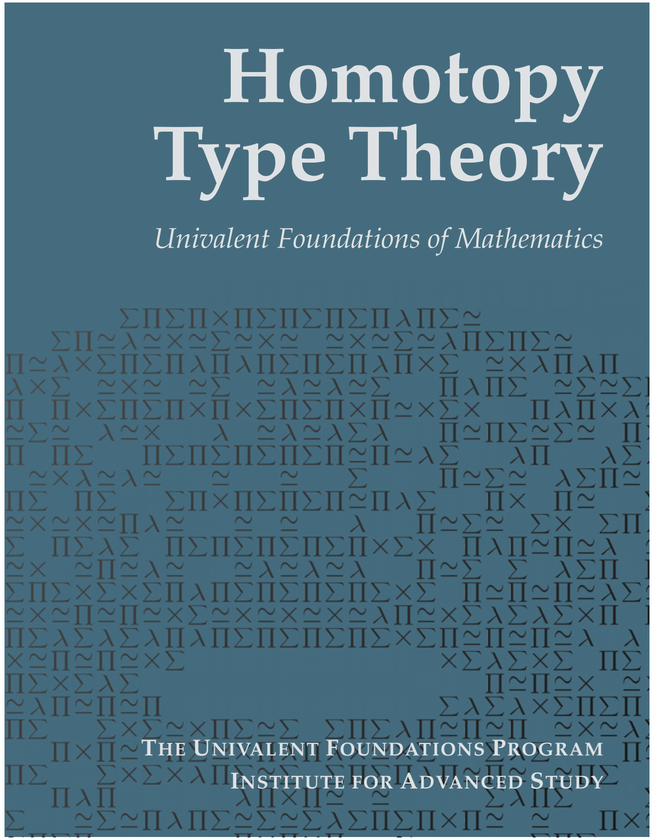
The OG resource

The **HoTT Book** was written during the *Special Year on Univalent Foundations of Mathematics* held in 2012–2013 at the Institute for Advanced Study, organized by Steve Awodey, Thierry Coquand, and Vladimir Voevodsky.

https://homotopytypetheory.org/book/

From the preface:

> *We did not set out to write a book. The present work has its origins in our collective attempts to develop a new style of "informal type theory" that can be read and understood by a human being, as a complement to a formal proof that can be checked by a machine.*



New resources I

Introduction to Univalent Foundations of Mathematics with Agda (2019; last updated: 2025)

https://martinescardo.github.io/HoTT-UF-in-Agda-Lecture-Notes/index.html

Written *in* the proof assistant Agda!

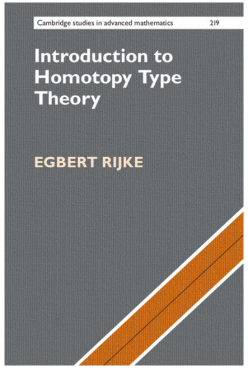
Martín Escardó

Introduction to Homotopy Type Theory (2022)

arXiv:2212.11082 (free)
Published by Cambridge University Press
doi:10.1017/9781108933568 (£50)

Largely formalized in agda-unimath.

Beginner friendly; modern perspective on HoTT.



Egbert Rijke

New resources II

HoTTEST Summer School 2022

https://hottest-seminar.github.io/HoTTESTSummerSchool.html

- The school featured two tracks: one on **Agda** and one on **HoTT**.
  - Each track had 12 lectures and as many problem sessions, all available on YouTube: https://tinyurl.com/HoTTEST-Summer-School-YT
  - The HoTT track was based on Rijke's book.
- There is an associated GitHub repository: https://github.com/martinescardo/HoTTEST-Summer-School
- You can still join the **Discord** to discuss, ask questions, etc. — https://discord.gg/tkhJ9zCGs9

New resources II

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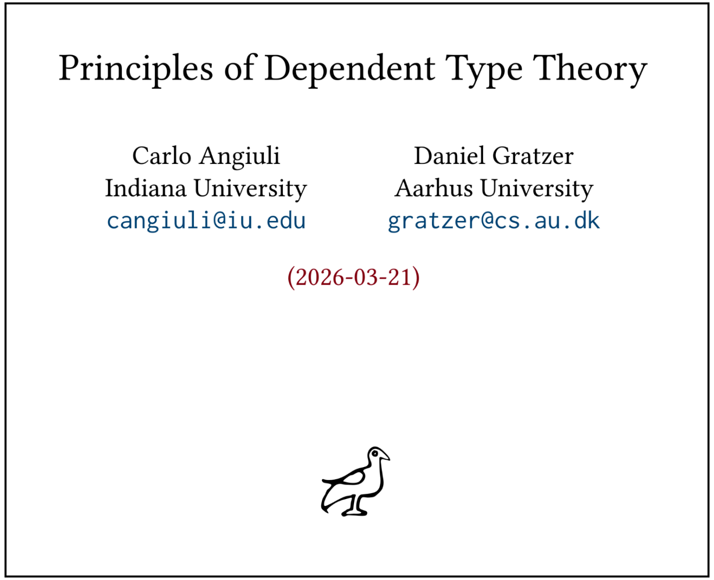
The HoTT Book covers more synthetic homotopy theory and (univalent) category theory than Rijke's book and Escardó's notes, but **no work is a subset of any of the others**.

What about cubical type theory, metatheory, semantics...?


Principles of Dependent Type Theory

Carlo Angiuli
Indiana University
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Daniel Gratzer
Aarhus University
gratzer@cs.au.dk

(2026-03-21)


Daniel Gratzer


Carlo Angiuli

https://www.danielgratzer.com/papers/type-theory-book.pdf

To be published by Cambridge University Press

Metatheory and implementation	§3
Semantics	§6 (draft)
Cubical type theory	§5.3 (draft)