# Introduction to homotopy theory type / univalent foundations

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# Overview

1. A few words on constructivism
2. Introduction to some fundamental notions from Voevodsky's univalent foundations
3. Very brief introduction to the syntax of MLTT

# Constructivism and computation

# Martin-Löf on mathematics and computation

From Per Martin-Löf's 1982 Constructive Mathematics and Computer Programming:
https://archive-pml.github.io/martin-lof/pdfs/Constructive-mathematics-and-computer-programming-1982.pdf

*[...] the close connection between constructive mathematics and programming explains why the intuitionistic type theory (Martin-Löf 1975), which I began to develop solely with the philosophical motive of clarifying the syntax and semantics of intuitionistic mathematics, may equally well be viewed as a programming language.*

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**Curry–Howard–Lambek correspondence**:

- the proof of $P \to (Q \to P)$ corresponds to the program that takes two inputs and returns only the first; in λ-calculus notation $\lambda p.\lambda q.p$.
- the inference rule $\dfrac{P \to R \qquad Q \to R}{P \vee Q \to R}$ corresponds to a case analysis program.

In the concluding remarks of that paper, Martin-Löf writes:

*[the] uninhibited use [of excluded middle] would lead to programs that you did not know how to execute.*

Indeed, we can add excluded middle, $P \vee \neg P$, as an axiom to MLTT, but how would we compute with it?

# Why cubical type theory?

- By Voevodsky's simplicial sets model, we know that we can consistently add his univalence axiom to MLTT.

  But just like with excluded middle, that does not give it a computational interpretation.

  In particular, proofs that use univalence may get "stuck" on it when we try to compute with them.

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- Coquand et al. created **cubical type theory** to give a computational meaning to the univalence axiom: in cubical type theory there is a program that implements univalence.

- Cubical type theory was inspired by a **constructive** model in cubical (as opposed to simplicial) sets, so the (constructive) semantics came first.

Thierry Coquand

# HoTT and constructive/classical mathematics

By default, HoTT is constructive in the sense that excluded middle or the axiom of the choice are not baked in. However, as detailed on the next slide,

**HoTT is compatible with classical axioms** (provided they are phrased carefully).

*Question*: So why is there often a focus on constructive mathematics in HoTT?

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By default, HoTT is constructive in the sense that excluded middle or the axiom of the choice are not baked in. However, as detailed on the next slide,

**HoTT is compatible with classical axioms** (provided they are phrased carefully).

*Question*: So why is there often a focus on constructive mathematics in HoTT?

*Some answers:*

1. For historical/social reasons: type theory originated as a system for constructive mathematics.
2. Because of the connection to programming, an experience strenghtened / (best?) felt by formalizing mathematics (in Agda).
3. Because of the synthetic mathematics programme: even if you "believe in" excluded middle and the axiom of choice, these principles fail in some toposes.
   *This is not a matter of philosophy/taste but of mathematics!*

# Voevodsky's univalent foundations

# Voevodsky's univalent foundations

The simplicial sets model of HoTT is very relevant and important, but I focus on the novelties of Voevodsky's *Foundations* Coq library here.

The significance lied in his **new and elegant definitions**, among them:

- $n$-(truncated) types (called "hlevels" by Voevodsky), with **contractible** types as a fundamental instance;
- the **fiber** of a function;
- a well behaved notion of **equivalence**;
- the **univalence axiom** which connects equivalences with Martin-Löf's identity types.

## $n$-types

Voevodsky realized that MLTT's identity types can be used to **define** (higher) groupoids *inside type theory*.

For $n \in \{-2, -1, 0, \ldots\}$, we can define a hierarchy of $n$-types (a.k.a. $n$-truncated types)

$$-2\text{-types} \subset -1\text{-types} \subset 0\text{-types} \subset \ldots \qquad \text{with}$$

- $0$-types being ordinary sets ("types à la Axiom K");
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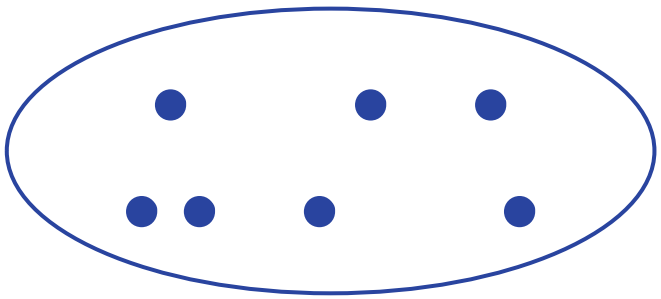
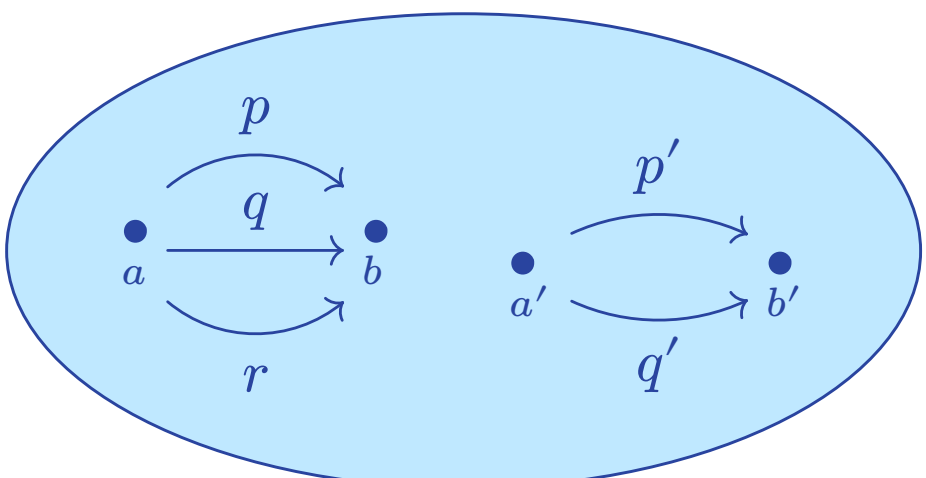
- $(-1)$-types are the **propositions** (or **subsingletons**) — this is where we will **recover logic**;
- $(-2)$-types have just a single point.
  These are known as **contractible** — this notion is surprisingly fundamental!

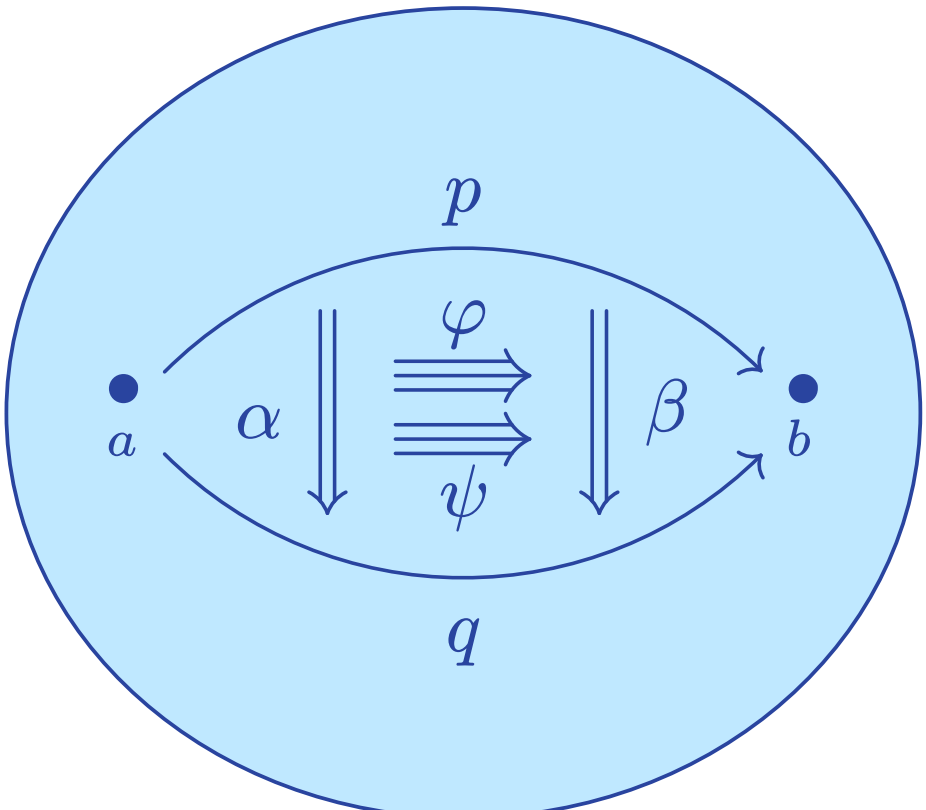
For $n \geq 0$, a good way to think of $n$-types is as those types whose (iterated) identity types trivialize above level $n$.



An illustration of a 0-type.




An illustration of a 1-type.




An illustration of a higher type.

# Examples of (higher) types

- A **proposition**, **subsingleton**, or **$-1$-type** is a type with at most one element.
- A **$0$-type** or **set** is a type whose identity types all have at most one element, i.e. all its identity types are propositions.
- An **$n+1$-type** is a type whose identity types are all $n$-types.

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- The empty type $\mathbb{0}$ and the type $\mathbb{1}$ with exactly one element are examples of propositions, i.e. $-1$-types.
- We assume to have an inductive type $\mathbb{N}$ of natural numbers. It can be shown to be a $0$-type.

  We can construct $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ as usual, and the resulting types are $0$-types.
- With **univalence**, we can construct examples of $n$-types for any $n$, and even of untruncated types (i.e. types with nontrivial identity types at any level).

# Function extensionality (warm up to the univalence axiom)

**Question:** When should two functions be equal?

**Answer:** Two functions $f, g : A \to B$ should be equal when they are pointwise equal, i.e. for every point $a$ in $A$, the values $f(a)$ and $g(a)$ are equal.

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- Suppose we have functions $f, g : A \to B$ in MLTT. We can construct a type $f \sim g$ expressing that $f$ and $g$ are pointwise equal.
- We can also consider the identity type $\mathrm{Id}_{A \to B}(f, g)$. How does it compare to $f \sim g$?

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- We can also consider the identity type $\mathrm{Id}_{A\to B}(f, g)$. How does it compare to $f \sim g$?
- We always have $\mathrm{Id}_{A\to B}(f, g) \longrightarrow f \sim g$, but the converse is not derivable in plain MLTT.
- Therefore, people considered the **function extensionality** axiom:

$$f \sim g \longrightarrow \mathrm{Id}_{A\to B}(f, g) \quad \text{“pointwise equal functions are equal”.}$$

# The univalence axiom: universe extensionality

**Question:** When should two types, living in a universe $\mathcal{U}$, be equal?

**Voevodsky's answer:** Two such types $A$ and $B$ should be equal when they are **equivalent**.

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- This requires a good notion of equivalence! Fortunately, Voevodsky provided this; he defined what it means for a function $f : A \to B$ to be an equivalence.
  Voevodsky: $f$ is an equivalence when its fibers are contractible types. Alternatively, when it has a left and right inverse.
- We can collect all equivalences from $A$ to $B$ into a type $A \simeq B$.
- A naive formulation of the above answer would be to assume that there is a map $A \simeq B \longrightarrow \mathrm{Id}_{\mathcal{U}}(A, B)$.

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- A naive formulation of the above answer would be to assume that there is a map $A \simeq B \longrightarrow \mathrm{Id}_{\mathcal{U}}(A, B)$.
- A more refined formulation, **Voevodsky's univalence axiom**, says that the canonical map $\mathrm{Id}_{\mathcal{U}}(A, B) \longrightarrow A \simeq B$ which can always be defined is itself an equivalence.

# Univalence at work: examples of higher types

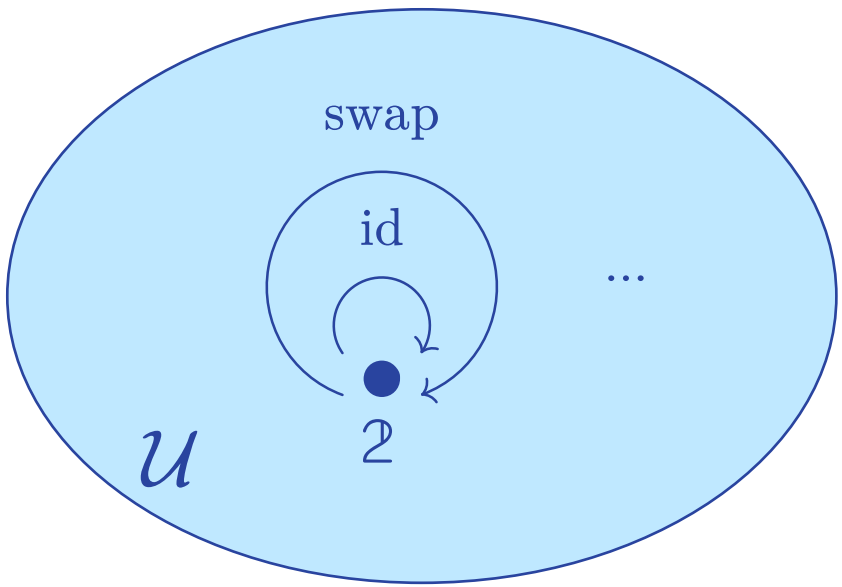
## Universes

Let $2$ be the type with exactly two elements ($0$ and $1$, say) and let $\mathcal{U}$ be a universe that contains it.

If $\mathcal{U}$ is univalent, then $\mathrm{Id}_{\mathcal{U}}(2, 2)$ and $2 \simeq 2$ are equivalent.

But it's easy to see that there exactly two equivalences from $2$ to itself, namely the identity and the map that swaps $0$ and $1$.

Thus, the identity type $\mathrm{Id}_{\mathcal{U}}(2, 2)$ has exactly two elements, and $\mathcal{U}$ is an example of a higher type.

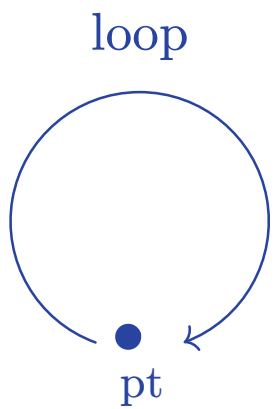


## Circle

The circle is an example of a **higher inductive type**.

While the type $\mathbb{N}$ of natural numbers is inductively generated by two constructors $0 : \mathbb{N}$ and $\mathrm{succ}(n) : \mathbb{N}$ (for $n : \mathbb{N}$), the circle $S^1$ is inductively generated by

- a single point constructor $\mathrm{pt} : S^1$,
- and a *path* constructor $\mathrm{loop} : \mathrm{Id}_{S^1}(\mathrm{pt}, \mathrm{pt})$.




With univalence, we can show that $\mathrm{Id}_{S^1}(\mathrm{pt}, \mathrm{pt})$ is equivalent to $\mathbb{Z}$, the type of integers.

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We can similarly introduce higher spheres (via iterated suspensions, in fact).

# Univalence at work: isomorphic structures are equal

Pick your favourite mathematical structure, e.g. boolean algebras, topological spaces, groups, etc. For concreteness, I will pick groups here.

- We can define a type $\mathrm{Group}$ of groups.
- For groups, i.e. elements of this type $G, H : \mathrm{Group}$, we can consider $\mathrm{Id}_{\mathrm{Group}}(G, H)$.

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- It is a consequence of the univalence axiom that two groups are equal* if and only if the groups $G$ and $H$ are isomorphic.

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- If we have an element $p$ of $\mathrm{Id}_{\mathrm{Group}}(G, H)$, then *any* true statement about $G$ must also be true of $H$, since we can transport the proof of the statement for $G$ along $p$.

So with univalence, isomorphic groups **really** do behave the same.

# The syntax of MLTT

# Quick overview

| Type (former) | Elements/terms | Set theoretic anal. of the type |
|---|---|---|
| Empty type $\mathbb{0}$ | none | $\varnothing$ |
| Unit type $\mathbb{1}$ | $*$ | singleton set |
| Product type $A \times B$ | $(a, b)$ with $a : A$ and $b : B$ | cartesian product |
| Coproduct type $A + B$ | $\mathrm{inl}(a)$ with $a : A$ and $\mathrm{inr}(b)$ with $b : B$ | disjoint union |
| Dependent pair type $\sum(x : A), B(x)$ | $(a, b)$ with $a : A$ and $b : B(a)$ | $\coprod_{x \in A} B(x)$ |
| Function type $A \to B$ | $f$ such that for $a : A$, we get $f(a) : B$ | total, single-valued relations on $A$ and $B$ |
| Dependent function type $\prod(x : A), B(x)$ | $f$ such that for $a : A$, we get $f(a) : B(a)$ | set of sections of the projection $\left(\coprod_{x \in A} B(x)\right) \to A$ |

| Type (former) | Elements/terms | Set theoretic anal. of the type |
|---|---|---|
| Identity type $\mathrm{Id}_A(a,b)$ for a $0$-type $A$ | "proof that $a$ is equal to $b$" | equality of FOL |
| Identity type $\mathrm{Id}_A(a,b)$ for a general type $A$ | "path from $a$ to $b$ in $A$" | none (not directly anyway) |

From now on, instead of $\mathrm{Id}_A(a,b)$, I write $a =_A b$ or even just $a = b$ when $A$ can be inferred.

# Using type theory to encode mathematics

Recall that a **monoid** is a set $M$ with an associative binary operation $\cdot$ such that we have a *neutral* element $e \in M$, i.e. $m \cdot e = e \cdot m = m$ for all $m \in M$.

We thus (naively) define **the type of monoids** as

$$\begin{aligned}\sum(M : \mathcal{U}), \sum(- \cdot - : M \times M \to M), \sum(e : M),& \\ \left(\prod(m : M), (m \cdot e = m) \times (e \cdot m = m)\right)& \\ \times \left(\prod(x, y, z : M), (x \cdot y) \cdot z = x \cdot (y \cdot z)\right).&\end{aligned}$$

An element of this type is a tuple $(M, \cdot, e, \nu, \alpha)$ where $M$ is the carrier, $\cdot$ is the monoid operation, $e$ is the neutral element, and $\nu$ and $\alpha$ resp. witness neutrality and associativity.

Notice the use of dependent types!

# A refinement of the monoid example

- The theory of monoids is *not* about higher dimensional structures.

  Therefore, in HoTT, we futher add the condition

  $$\text{is-0-type}(M)$$

  to the earlier (preliminary) type of monoids.

- This addition also ensures that $\nu$ and $\alpha$ are elements of propositions ($-1$-types) and can be thought of as **logical proofs**, rather than *additional pieces of data*.
- We will return to the distinction between data and property in tomorrow's lecture.