On epimorphisms and acyclic types in univalent type theory

Ulrik Buchholtz[1] Tom de Jong[1] Egbert Rijke[2]

[1]University of Nottingham, UK
[2]University of Ljubljana, Slovenia

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Starting question

- Exercise in category theory:
  The epimorphisms of sets are precisely the surjections.

- **Question**:
  What are the epimorphisms of types?

- We answer this question in homotopy type theory (HoTT), where we have higher types.

Motivation for studying epimorphisms

- Epimorphisms are useful because

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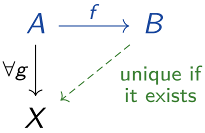

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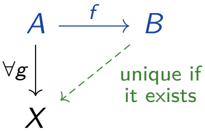
f is an epi $\iff$ 

- We show that epis of types are closely related to acyclic types.

  Classically, acyclic spaces are used in algebraic topology in
  - Quillen's plus construction,
  - the Kan–Thurston theorem, and
  - the Barratt–Priddy(–Quillen) theorem.

  So this leads to interesting synthetic homotopy theory!

Outline

1. Homotopy type theory (HoTT)
2. Epimorphisms in HoTT
3. A surjection that isn't an epimorphism
4. Characterization of epimorphisms
5. Example of an epimorphism

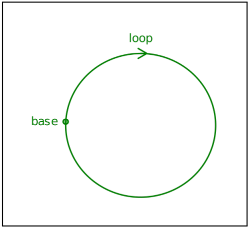
Homotopy type theory (HoTT)

- In HoTT, we think of types as spaces.
- If we have a type A with points $a, b : A$, then we may have identifications $p, q : a =_A b$ and higher identifications $\alpha, \beta : p =_{a =_A b} q$, etc.
- A type is a set or 0-type if there are no higher identifications. E.g. $\mathbb{N}$, $\mathbb{N} \to \mathbf{2}$, $\mathbb{N} \to \mathbb{N}$, etc. are all 0-types.

Higher types

- The circle $\mathbb{S}^1$

Higher inductive type
$\mathsf{base} : \mathbb{S}^1$
$\mathsf{loop} : \mathsf{base} = \mathsf{base}$



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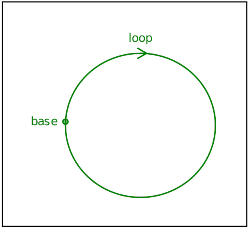

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- Similarly, we get the notion of a k-type for $k \geq 0$.
(Actually, $k \geq -2$.)

Epimorphisms in HoTT

- In 1-category theory, a morphism $f : A \to B$ is an epi(morphism) if for every object C and all morphisms $g, h : B \to C$, we have

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- Def. A map $f : A \to B$ is an epi if the canonical map

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  Note: C may be a *higher* type!

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and the canonical map

$$\begin{array}{ccccccc} \mathbb{Z} & \simeq & (g = g) & \longrightarrow & (g \circ f = g \circ f) & \simeq & \mathbb{Z}^2 \\ k & & & \longmapsto & & & (k, k) \end{array}$$

is not an equivalence.

Outline

- ✓ Homotopy type theory (HoTT)
- ✓ Epimorphisms in HoTT
- ✓ A surjection that isn't an epimorphism
- ⋆ Characterization of epimorphisms
- - Example of an epimorphism

Suspensions and acyclic types

- Def. The suspension ΣA of a type A is

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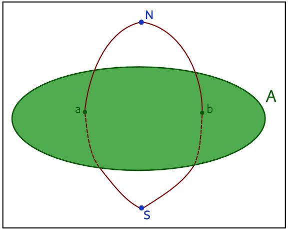

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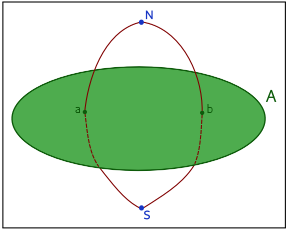

- Ex. The suspension of the circle is the sphere.
- Def. A type A is acyclic if $\Sigma A \simeq \mathbf{1}$.
- Ex. The unit type $\mathbf{1}$ is acyclic.

Characterization of epimorphisms

- Fact A map $f : X \to Y$ is epi w.r.t *sets* $\iff$ f is surjective.

- Surjectivity means: for every $y : Y$, the fiber of f is inhabited. That is, we have an element of the propositional truncation of

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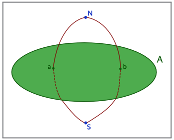
Theorem

A map $f : X \to Y$ is epi (w.r.t. all types) $\iff$ all fibers are acyclic.

That is, the suspension of $\mathrm{fib}_f(y)$ is equivalent to $\mathbf{1}$ for all $y : Y$.



Summarising the results presented so far

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An example of an epimorphism

- The theorem implies:

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- We'll present an illustrative example of an acyclic type.

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- We note:

  Thm. The only acyclic *set* is $\mathbf{1}$.

  So we look at higher types for acyclicity.

The Higman group, classically

- The Higman group H is defined as the group with 4 generators a, b, c, d and 4 relations

  $$r_a : a = [d, a] \quad r_b : b = [a, b] \quad r_c : c = [b, c] \quad r_d : d = [c, d],$$

  where $[x, y] \equiv xyx^{-1}y^{-1}$ denotes the commutator.

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  We build a space (CW-complex) with fundamental group H.

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- Classically, its classifying space is acyclic.

  We build a space (CW-complex) with fundamental group H.

- The group can be shown to be nontrivial, but it requires combinatorial group theory.

  For ≤ 3 generators and relations the presentation yields the trivial group!

The Higman group, type-theoretically

- We present the Higman space $\mathsf{B}H$ as the following higher inductive type:

$$\mathsf{pt} : \mathsf{B}H$$
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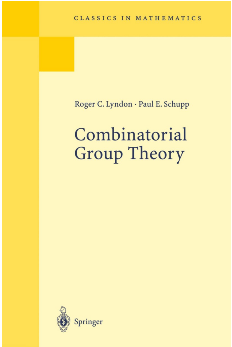
- For acyclicity, the suspension $\Sigma\,\mathsf{B}H$ of $\mathsf{B}H$ is contractible by:
  - a higher inductive presentation of $\Sigma\,\mathsf{B}H$, and
  - the Eckmann–Hilton argument: all higher homotopy groups are commutative.

Nontriviality of the Higman HIT

- We *completely avoid* classical combinatorial group theory in proving that the Higman HIT is nontrivial.



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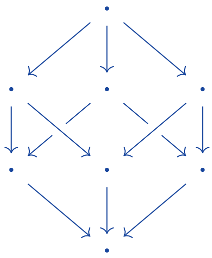
    If the bottom and top squares of a commutative cube are pushouts and the back sides are pullbacks,



    then the front sides are pullbacks too.

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- We can (re)construct $\mathrm{B}H$ as a series of such pushout squares.
- It also follows that $\mathrm{B}H$ is a 1-type: no need to truncate!

Summary

At higher types, the notion of epimorphism

- becomes quite strong,
- coincides with the notion of an acyclic map, and
- is interesting from the p.o.v. of synthetic homotopy theory.

Additional and future work

- Do the acyclic maps form an accessible modality?

- Many properties seem to need an additional axiom:

  > Plus Principle: Every acyclic and simply connected type is contractible.

  It follows from Whitehead's Principle (WP) and was highlighted by Hoyois in ∞-topos theory.

- We believe that plus-constructions can be performed in HoTT assuming WP, Sets Cover, and Countable Choice.

- Use the theory of binate groups to prove acyclicity of some infinitely presented groups?

- We also study k-epimorphisms and k-acyclic types.
  (Similar to k-equivalences and k-connected maps.)

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