The Leibniz adjunction in homotopy type theory, with an application to simplicial type theory

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Introduction & main results (informally)

- Riehl and Shulman introduced simplicial type theory (STT) as a synthetic framework for higher categories.[1]
- In homotopy type theory (HoTT) types behave like ∞-groupoids.
  STT additionally equips types with a notion of directed morphisms, via an interval.

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- Segal types are those types in which such morphisms compose.
- Composition should be associative, identities should be neutral w.r.t. composition, and higher coherences should be satisfied, e.g. the different ways of rewriting $\mathrm{id} \circ (f \circ \mathrm{id})$ to f should be the same, up to a higher morphism.

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🏁 For Segal types, all higher coherences can be derived.

🏁 We give a slick proof using some category theory and families instead of maps.

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Foundational setup

- Instead of simplicial type theory à la Riehl & Shulman, we follow Gratzer, Weinberger and Buchholtz[2] and work in an axiomatic extension of HoTT:

$$\text{HoTT} + \text{bounded distributive lattice } (I, 0, 1, \wedge, \vee).$$

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- Use I to construct the objects necessary for simplicial combinatorics (next slide).

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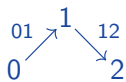
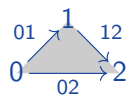
First main result (more precisely)

- Simplicial combinatorics: composition in X is given by extending along the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$.

$\Lambda^2_1 =$ 

$\Delta^2 =$ 

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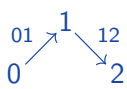
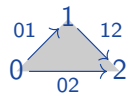
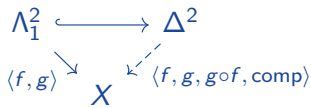
$\Lambda^2_1 =$ 

$\Delta^2 =$ 



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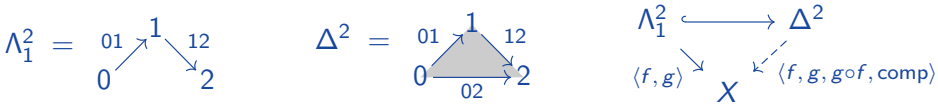

Def. A type X is a Segal type if every map $\Lambda^2_1 \to X$ has a unique (in the contractible/HoTT sense) extension to a map $\Delta^2 \to X$.

We also say that a Segal type has unique fillers for (2,1)-horns.

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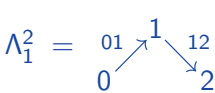

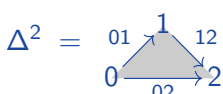

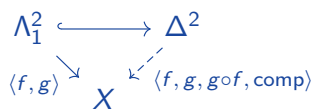

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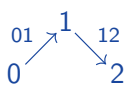
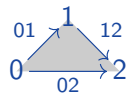
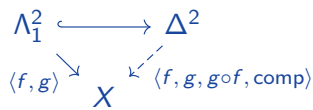
$\Lambda^2_1 =$ 

$\Delta^2 =$ 



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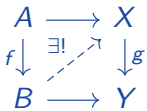
Thm. Any Segal type has unique fillers for all (n,k)-horns where $0 < k < n$.

- This generalizes a result of Riehl & Shulman for $n = 3$ and $k \in \{1, 2\}$, and is a STT version of a result by Lurie.

Reduction to the Leibniz adjunction

- Segal types can be characterized by orthogonality



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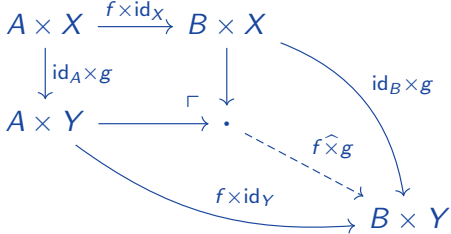
What we did:

- Reduce simplicial comb. to formal arguments about $\pitchfork$ and $\mathbin{\widehat{\times}}$ as much as possible.
- Prove the Leibniz adjunction for maps in HoTT.

Pushout-product and pullback-hom

Given $f : A \to B$ and $g : X \to Y$, we define



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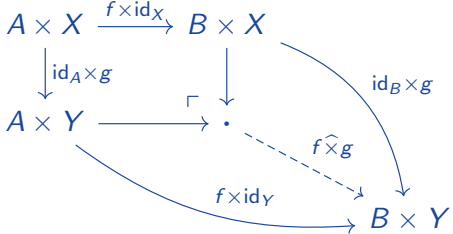

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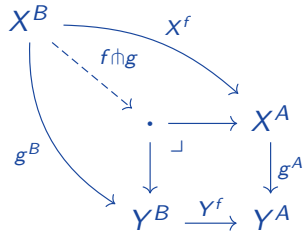

Second main result: the Leibniz adjunction in HoTT

Def. For $f : A \to B$ and $g : X \to Y$, write

$$\mathrm{Map}(f, g) :\equiv \left\{ \begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle f}\downarrow & \swarrow^{\alpha} & \downarrow{\scriptstyle g} \\ B & \xrightarrow{v} & Y \end{array} \right\}$$

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Thm. We have a natural equivalence

$$\mathrm{Map}(f \mathbin{\widehat{\times}} g, h) \simeq \mathrm{Map}(f, g \pitchfork h).$$

Towards a proof of the Leibniz adjunction

- Unfolding definitions, an element of $\mathrm{Map}(f \mathbin{\widehat{\times}} g, h)$ is a 7-tuple: three maps, three equations between maps and one additional coherence.

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- The above equivalence χ extends to an equivalence

$$\mathsf{Map}(f, g) \simeq \mathsf{Fam}(\chi\, f, \chi\, g).$$

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   Calculation is guided by the desired equations
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- Steps (1)–(3) turn out to be relatively easy and natural. ☺
  E.g. $(A, B) \mathbin{\widehat{\times}} (X, Y) \coloneqq (A \times X, (a, x) \mapsto B\, a * Y\, x)$,
  where $*$ denotes the join of types.

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Fully formalized in Cubical Agda.

References

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