Epimorphisms and Acyclic Types

Talk

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Starting question

- Exercise in category theory:
  *The epimorphisms of **sets** are precisely the surjections.*

- **Question**:
  *What are the epimorphisms of **types**?*

- We study this in homotopy type theory (HoTT) which has **higher types**.

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- Exercise in category theory:
  The epimorphisms of ***sets*** *are precisely the surjections.*

- **Question**:
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- We study this in homotopy type theory (HoTT) which has **higher types**.

Epimorphisms in HoTT are precisely the fiberwise acyclic maps.

Classically, acyclic spaces are used in algebraic topology in

- Quillen's plus construction,
- the Kan–Thurston theorem, and
- the Barratt–Priddy(–Quillen) theorem.

So this leads to interesting synthetic homotopy theory!

Synthetic homotopy theory

- Everything we do in HoTT is automatically/necessarily invariant under homotopy.
- This is both a blessing (no need for: "up to...") and a curse as it means that some (point-set based) constructions are not (readily) available in HoTT.
- In practice this means we work with universal properties only.

Epimorphisms in HoTT

- In 1-category theory, a morphism $f : A \to B$ is an epimorphism if for all objects C and all morphisms $g, h : B \to C$, we have

$$(g \circ f = h \circ f) \iff (g = h).$$

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- To get a homotopically well behaved notion, we use the following in HoTT:

  Def. A map $f : A \to B$ is an epimorphism if the canonical map

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- Equivalently, for any $f' : A \to C$, the **type of extensions of** f' **along** f

  $$\sum_{g : B \to C} g \circ f = f'$$

  has at most one element. That is, the pair of the map **together with the commutativity witness** is unique, if it exists.

Suspensions and acyclic types

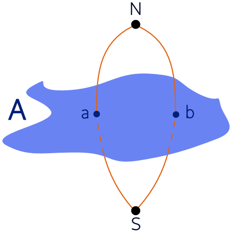
- Def. The suspension ΣA of a type A is the pushout

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow S \\ \mathbf{1} & \xrightarrow[N]{} & \Sigma A \end{array}$$

Higher inductive type (HIT)

$$\mathsf{N}, \mathsf{S} : \Sigma A$$

$$\mathsf{merid} : A \to (\mathsf{N} = \mathsf{S})$$



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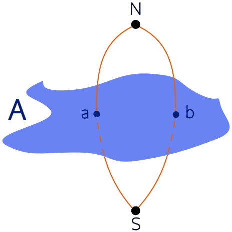

- Ex. The suspension of the circle is the sphere.

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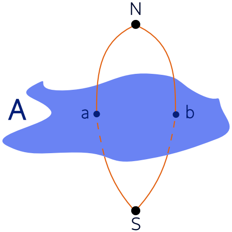

- Def. A type A is acyclic if ΣA is contractible, i.e. $\Sigma A \simeq \mathbf{1}$.

Characterization of epimorphisms

- Thm. A map $f : A \to B$ is an epimorphism if and only if each fiber $\sum_{a:A} f(a) = b$ is an acyclic type.

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- Ex. The unique map $f : \mathbf{2} \to \mathbf{1}$ is **not** an epimorphism: its fiber at $\star : \mathbf{1}$ is equivalent to $\mathbf{2}$ and $\Sigma\,\mathbf{2} \simeq \mathbb{S}^1$.

Examples of acyclic types

- Hatcher's 2-dimensional complex is an example of a **nontrivial acyclic** space.
  - We import Hatcher's 2-dimensional complex as a HIT X with constructors:

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  - The type $\mathsf{pt} =_X \mathsf{pt}$ has a surjection to the alternating group A_5, so X is nontrivial.
  - Acyclicity of X follows from a purely categorical/type-theoretic argument using **Eckmann-Hilton** (higher loop spaces are abelian).

Examples of acyclic types

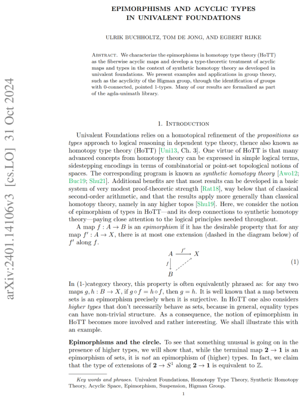
- Another example is the classifying type of Higman's group which is given by 4 generators and relations.
  - Acyclicity follows from Eckmann-Hilton again.
  - Showing that the group is nontrivial is hard and traditionally relies on **combinatorial group theory**.

    We use techniques from **higher category theory** instead:

descent for pushouts + results on 0-truncated maps and pushouts [Wärn]

Check out the paper!

- Our paper is available on `arXiv:2401.14106` and will appear in *The Journal of Symbolic Logic*.



EPIMORPHISMS AND ACYCLIC TYPES
IN UNIVALENT FOUNDATIONS

ULRIK BUCHHOLTZ, TOM DE JONG, AND EGBERT RIJKE

Abstract. We characterize the epimorphisms in homotopy type theory (HoTT) as the fiberwise acyclic maps and develop a type-theoretic treatment of acyclic maps and types in the context of synthetic homotopy theory as developed in univalent foundations. We present examples and applications in group theory, such as the acyclicity of the Higman group, through the identification of groups with 0-connected, pointed 1-types. Many of our results are formalized as part of the agda-unimath library.

1. Introduction

Univalent Foundations relies on a homotopical refinement of the *propositions as types* approach to logical reasoning in dependent type theory, thence also known as homotopy type theory (HoTT) [Uni13, Ch. 3]. One virtue of HoTT is that many advanced concepts from homotopy theory can be expressed in simple logical terms, sidestepping encodings in terms of combinatorial or point-set topological notions of spaces. The corresponding program is known as *synthetic homotopy theory* [Awo12; Buc19; Shu21]. Additional benefits are that most results can be developed in a basic system of very modest proof-theoretic strength [Rat18], way below that of classical second-order arithmetic, and that the results apply more generally than classical homotopy theory, namely in any higher topos [Shu19]. Here, we consider the notion of epimorphism of types in HoTT—and its deep connections to synthetic homotopy theory—paying close attention to the logical principles needed throughout.

A map $f : A \to B$ is an *epimorphism* if it has the desirable property that for any map $f' : A \to X$, there is at most one extension (dashed in the diagram below) of f' along f.

$$A \xrightarrow{f'} X, \quad f : A \to B, \quad B \dashrightarrow X \tag{1}$$

In (1-)category theory, this property is often equivalently phrased as: for any two maps $g, h : B \to X$, if $g \circ f = h \circ f$, then $g = h$. It is well known that a map between sets is an epimorphism precisely when it is surjective. In HoTT one also considers *higher types* that don't necessarily behave as sets, because in general, equality types can have non-trivial structure. As a consequence, the notion of epimorphism in HoTT becomes more involved and rather interesting. We shall illustrate this with an example.

Epimorphisms and the circle. To see that something unusual is going on in the presence of higher types, we will show that, while the terminal map $\mathbf{2} \to \mathbf{1}$ is an epimorphism of sets, it is *not* an epimorphism of (higher) types. In fact, we claim that the type of extensions of $\mathbf{2} \to S^1$ along $\mathbf{2} \to \mathbf{1}$ is equivalent to $\mathbb{Z}$.

Key words and phrases. Univalent Foundations, Homotopy Type Theory, Synthetic Homotopy Theory, Acyclic Space, Epimorphism, Suspension, Higman Group.

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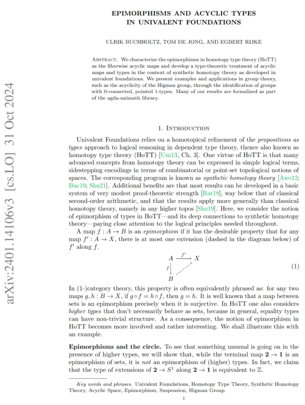

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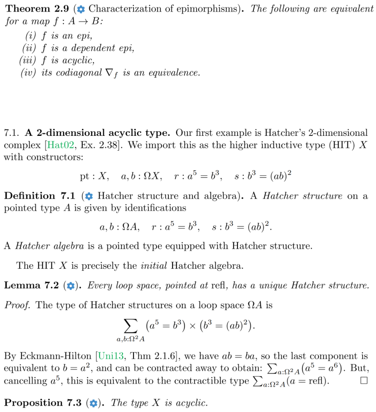

Theorem 2.9 (⚙ Characterization of epimorphisms). *The following are equivalent for a map $f : A \to B$:*

(i) f is an epi,
(ii) f is a dependent epi,
(iii) f is acyclic,
(iv) its codiagonal ∇_f is an equivalence.

7.1. **A 2-dimensional acyclic type.** Our first example is Hatcher's 2-dimensional complex [Hat02, Ex. 2.38]. We import this as the higher inductive type (HIT) X with constructors:

$$\mathrm{pt} : X, \quad a, b : \Omega X, \quad r : a^5 = b^3, \quad s : b^3 = (ab)^2$$

Definition 7.1 (⚙ Hatcher structure and algebra). A *Hatcher structure* on a pointed type A is given by identifications

$$a, b : \Omega A, \quad r : a^5 = b^3, \quad s : b^3 = (ab)^2.$$

A *Hatcher algebra* is a pointed type equipped with Hatcher structure.

The HIT X is precisely the *initial* Hatcher algebra.

Lemma 7.2 (⚙). *Every loop space, pointed at* refl, *has a unique Hatcher structure.*

Proof. The type of Hatcher structures on a loop space ΩA is

$$\sum_{a,b:\Omega^2 A} (a^5 = b^3) \times (b^3 = (ab)^2).$$

By Eckmann-Hilton [Uni13, Thm 2.1.6], we have $ab = ba$, so the last component is equivalent to $b = a^2$, and can be contracted away to obtain: $\sum_{a:\Omega^2 A}(a^5 = a^6)$. But, cancelling a^5, this is equivalent to the contractible type $\sum_{a:\Omega^2 A}(a = \mathrm{refl})$. □

Proposition 7.3 (⚙). *The type X is acyclic.*

- Many of its results are formalized in the proof assistant **Agda**. Clicking a ⚙ next to a definition, lemma, theorem, etc. in the paper takes you to its formalization.

Future work

- Do the acyclic maps form an accessible modality?

- Some properties of acyclic maps seem to need an additional axiom:

Plus Principle: Every acyclic and simply connected type is contractible.

  It follows from **Whitehead's Principle** and was highlighted by Hoyois in the context of ∞-toposes.

  Whitehead fails in the ∞-topos of parametrized spectra which *does* validate the Plus Principle [Anel]. Is there an ∞-topos where the Plus Principle fails?

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`arXiv:2401.14106`

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