ACYCLIC TYPES AND THE PLUS PRINCIPLE, REVISITED

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Building on [1], we record a few additional observations regarding acyclic types and the plus principle.

1. Acyclic identity types are contractible

Proposition 1.1 Let A be a pointed type. The loop space ΩA is acyclic if and only if it is contractible.

Proof In the nontrivial direction, suppose ΩA is acyclic. By the suspension-loop space adjunction and acyclicity, we have equivalences

$$\left(\Omega A \to_{\mathrm{pt}} \Omega A\right) \simeq \left(\Sigma \Omega A \to_{\mathrm{pt}} A\right) \simeq \left(\mathbb{1} \to_{\mathrm{pt}} A\right) \simeq \mathbb{1}.$$

Hence, the type of pointed endomaps on ΩA is contractible. In particular, the constant map at refl and the identity must be equal, so that any $p : \Omega A$ is equal to refl, making ΩA contractible. □

Corollary 1.2 Any acyclic identity type is contractible.

Proof Suppose $x =_A y$ is acyclic. Then it is 0-connected, so we may assume to have $p : x =_A y$. But then path composition with p induces an equivalence between $x =_A y$ and $\Omega(A, x)$, so that Proposition 1.1 applies. □

2. Statements equivalent to the plus principle (Section 4 of [1])

For completeness, we recall the following principle from [1].

Definition 2.3 *(Plus principle)* The **plus principle**, abbreviated (PP), asserts that any simply connected acyclic type is contractible.

In [1], we showed that (PP) implies that acyclic 1-equivalences are equivalences [1, **Lemma 4.4**]. The converse holds too, trivially: if A is a simply connected acyclic type, then $A \to \mathbb{1}$ is an acyclic 1-equivalence and it's an equivalence if and only if A is contractible.

Let $f' : A \to X$ be any map and $f : A \to B$ be acyclic. In [1], **Proposition 4.5** showed that (PP) implies that f' extends along f if and only if we have the inclusion $\ker(\pi_1(f)) \subseteq \ker(\pi_1(f'))$ (for any choice of base point in A).

The converse holds too, as we show now. Let A be a simply-connected acyclic type. Note that A is contractible if and only if the identity on A extends along the unique map $f : A \to \mathbb{1}$. By assumption it suffices to show that $\ker(\pi_1(f)) \subseteq \ker(\pi_1(\mathrm{id}_A))$, but $\ker(\pi_1(f)) = \pi_1(A)$ which is trivial as A is simply connected.

3. Hypoabelian types, lifting, and the plus principle (Section 5 of [1])

For completeness, we recall the following definition from [1].

Definition 3.4 *(Hypoabelian)* A type X is **hypoabelian** if, for every $x : X$, every perfect subgroup of $\pi_1(X, x)$ is trivial. We extend this notion to maps fiberwise.

In [1], **Lemma 5.2** showed that (PP) implies that acyclic types are orthogonal to hypoabelian ones. In particular, every type that is both hypoabelian and acyclic must be contractible. Since simply connected types are (trivially) hypoabelian, this shows the following.

Proposition 3.5 The plus principle is equivalent to the assertion that every hypoabelian acyclic type is contractible. □

Proposition 3.6 If acyclic maps enjoy the left lifting property against a class of morphisms that include the simply connected maps, then (PP) holds.

Proof Let A be a simply connected acyclic type. By assumption there exists a lift

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d] & A \arrow[d] \\ \mathbb{1} \arrow[r, "\mathrm{id}_{\mathbb{1}}"'] \arrow[ur, dashed] & \mathbb{1} \end{tikzcd}$$

showing that A must be contractible. □

4. An extension criterion in the absence of the plus principle

Theorem 4.7 Let $f : A \to B$ be an acyclic map and $f' : A \to X$ be arbitrary. Then f' extends along f (necessarily uniquely) if and only if for every $a, a' : A$ we have a map from $f(a) = f(a')$ to $f'(a) = f'(a')$.

Note that since extensions along acyclic maps are unique if they exist, we can truncate the condition in the theorem, i.e. we can consider

$$\left\| \prod (a, a' : A), f(a) = f(a') \to f'(a) = f'(a') \right\|$$

instead.

Proof The condition is necessary: if h is an extension of f' along f and we have $f(a) = f(a')$, then $f'(a) = h(f(a)) = h(f(a')) = f'(a')$.

To see that it is sufficient, we first form the pushout

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "g'"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

just like in the proof of [1, Proposition 4.5]. Since f is acyclic it is in particular surjective. Moreover, f is balanced [1, Theorem 2.26], so that the pullback

$$\begin{tikzcd} A \times_B A \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is also a pushout. Hence by pushout pasting, the square

$$\begin{tikzcd}
A \times_B A \arrow[r, "f' \circ \mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "g"] \\
A \arrow[r, "g' \circ f"'] & P \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is again a pushout. The condition of the theorem yields an induced (dashed) map

$$\begin{tikzcd}
A \times_B A \arrow[r, "f' \circ \mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X \arrow[d, "g"] \arrow[ddr, bend left, "\mathrm{id}_X"] & \\
A \arrow[r, "g' \circ f"'] \arrow[drr, bend right, "f'"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "r"] & \\
 & & X
\end{tikzcd}$$

But now the desired extension is given by $B \xrightarrow{g'} P \xrightarrow{r} X$.

(Moreover, g is actually an equivalence, as it is acyclic (it is a pushout of f) and any acyclic map with a retraction is an equivalence.) □

Corollary 4.8 For an acyclic type A and any type X, the map that takes $x : X$ to the function $a \mapsto x$ is an equivalence when the codomain is restricted to functions that are truncatedly weakly constant:

$$X \simeq \left(\sum(f : A \to X), \|f \text{ is weakly constant}\|\right).$$

Proof Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "x \mapsto (\lambda a.\, x, \ldots)"] \arrow[dd, "\mathrm{id}_X"', "\simeq"] & \sum(f : A \to X), \|f \text{ is weakly constant}\| \arrow[d, "\simeq \text{ (by Theorem 4.7)}"] \\
 & \sum(f : A \to X), \sum(x : X), \|f \text{ is weakly constant}\| \times (f \sim \mathrm{const}_x) \arrow[d, "\simeq \text{ (leaving out a redundant prop.)}"] \\
X & \sum(f : A \to X), \sum(x : X), (f = \mathrm{const}_x) \arrow[l, "\simeq", "\mathrm{pr}_2"']
\end{tikzcd}$$

which commutes definitionally. By 3-for-2 of equivalences, the top map must be an equivalence. □

Corollary 4.9 For an acyclic map f, the following are logically equivalent:

(i) f is left cancellable;
(ii) $\|f$ is left cancellable$\|$;
(iii) f is an embedding;
(iv) f is an equivalence.

Proof Clearly (iv) implies all other conditions, so it suffices to show that (i) implies (iv). So suppose that $f : A \to B$ is acyclic and we have a dependent map

$\prod(a, a' : A), f(a) = f(a') \to a = a'$. By Theorem 4.7, the identity on A extends along f, yielding $g : B \to A$ with $g \circ f = \mathrm{id}_A$. Since f is epic, $f \circ g = \mathrm{id}_B$ also follows and f is indeed an equivalence. $\square$

4.1. **Additional idea(?) for Theorem 4.7.**

We can freely assume that f' is surjective (cf. the proof of [1, Proposition 4.5]), so that g' is also surjective (surjections are exactly (-1)-connected maps and those are pushout-stable). Hence, since g is balanced (as the pushout of an acyclic map), the inner pullback square

$$\begin{tikzcd}
A \arrow[rr, "f'"] \arrow[dd, "f"'] \arrow[dr, dashed] & & X \arrow[dd, "g"] \\
& B \times_P X \arrow[ur, "\mathrm{pr}_2"] \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & \\
B \arrow[rr, dash, "g'"] & & P
\end{tikzcd}$$

is also a pushout (as is the outer square).

Can we somehow use this to show that g must be an equivalence?

References

1. Buchholtz, U., de Jong, T., Rijke, E.: Epimorphisms and Acyclic Types in Univalent Foundations. The Journal of Symbolic Logic. 1–36 (2025). `https://doi.org/10.1017/jsl.2024.76`

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